Flight-to-Safety in a New Keynesian Model*

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Abstract

This paper builds a tractable New Keynesian model with idiosyncratic risk, incomplete markets, and nominal safe assets to study the transmission of uncertainty shocks through investors' portfolio decisions and how monetary-fiscal policy can stabilize fluctuations in the demand for safe assets. In response to a sudden increase in uncertainty, investors reallocate their resources from productive assets to safe assets for precautionary reasons. When prices are sticky, the real value of nominal safe assets cannot flexibly adjust. Instead, the adjustment pressure leads to an aggregate demand recession and undershooting in the price of productive capital assets. Conventional monetary policy that operates through interest rate changes alone has limited power in influencing household portfolios. Instead, fiscal policy plays a crucial role in price stabilization and optimal policy.

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1 Introduction

In recent years, variations in economic uncertainty have become an increasingly important source of aggregate fluctuations. In times of high uncertainty such as the Great Recession and the COVID-19 crisis, government debts (e.g., reserves and treasury bonds) provide nominal insurance against idiosyncratic risk. Following an abrupt rise in uncertainty, investors reallocate their resources from productive capital to safe government bonds for precautionary motives. Such "flight-to-safety" episodes feature sharp declines in the demand for capital, which leads to recessions in investment. Additionally, in an environment where short-term output is demand-determined, heightened uncertainty depresses effective demand and triggers recessions in consumption and output.

In this paper, we highlight a novel transmission mechanism of uncertainty shocks through portfolio reallocations. To this end, we build a tractable New Keynesian model with idiosyncratic risk and "flight-to-safety" behaviors. In our model, capital and government bonds are imperfect substitutes as they differ in their risk exposures. Capital is more productive but subject to idiosyncratic return risk, while government bonds are unproductive but nominally safe. As uncertainty about the return on capital increases, the private sector hoards more government debts as consumption insurance, which we refer to as "flight-to-safety."

Importantly, the real value of *nominal* government bonds is directly linked to the price level. In the presence of nominal rigidity, the value of existing government debts adjusts slowly to shocks through changes in the price level. Consequently, movements in demand for safe bonds are disconnected from the bond price in the short run. Since changes in the bond price cannot accommodate uncertainty-induced fluctuations in household portfolios, uncertainty shocks instead result in *overshooting* of capital values. In the New Keynesian model where short-term output is demand-determined, a decline in the capital price lowers household wealth, which initiates a general decline in aggregate economic activities.

We further show that our model generates realistic business cycle co-movements following uncertainty shocks. Specifically, our model predicts recessions in investment, consumption, and output, and disinflation after uncertainty rises, as have been identified by previous papers (e.g., Bachmann, Elstner and Sims, 2013; Basu and Bundick, 2017; Carriero, Clark and Marcellino, 2018). Further, following a positive uncertainty shock, the economy in our model exhibits both heightened risk premium and low output, an important disconnect highlighted by Caballero and Simsek (2021).

The second focus of this paper is on stabilization policies. To stabilize the economy after uncertainty shocks, the government needs to either expand the supply of safe assets or suppress the demand for safe assets. On the supply side, traditional monetary policy that operates through interest rate changes can only move the value of nominal government bonds slowly through (dis)inflation when the quantity of bonds is controlled separately by the fiscal authority. In contrast, the government can rapidly move the supply of safe assets using fiscal policy.

On the demand side, it is sometimes believed the central bank can push households into holding riskier assets simply by adjusting the risk-free rate. While it is true in New Keynesian models without safe assets (e.g., Basu and Bundick, 2017), we show that the result breaks down when safe assets are in a positive net supply. In particular, the effect of interest rate policy will depend crucially on the fiscal responses it induces. Behind this result is a simple but yet underappreciated intuition. Households' demand for safe assets depends on the level of uncertainty and returns on available assets. When interest rates drop, the interest payments on existing bonds fall simultaneously. Without supporting tax policies, interest rates cuts result in less bond issuance (decline in seigniorage income), which inflate the real return on bonds and offset the impact of interest rate policy on the real return on government bonds. We formalize this intuition by showing exactly how household portfolio decisions depend on (future) uncertainty and government policies.

To better understand the importance of the portfolio choice channel, we compare our model to an environment where government bonds are in zero net supply. Without having to fund interest payments on its debts, the government can freely vary the risk-free rate via monetary policy. Indeed, in this case, we recover the classical result that the central bank can stabilize the economy without any fiscal policy if the policy rate tracks the natural rate perfectly. Furthermore, even under the same monetary policy responses, the effects of uncertainty shocks are significantly amplified in the model with government bonds. The amplification result suggests that the overshooting mechanism has a major role in shaping macroeconomic dynamics following uncertainty shocks.

We then proceed to study how monetary-fiscal policy should be utilized to stabilize uncertainty-induced fluctuations in household portfolios. In response to uncertainty shocks, the fiscal authority can react by adjusting either lump-sum taxes or distortionary taxes.[1] While both types of policy can potentially stabilize the economy in the short run, they have distinct implications for welfare and longer-term economic outcomes. Importantly, lump-sum taxes are non-distortionary, and have the potential of ameliorating the impact of nominal rigidity without distorting long-run economic outcomes. Nevertheless, distortionary taxes have more profound impact on households' portfolio choices and can lead to different steady-state equilibria. Therefore, such taxes are more suitable for correcting incomplete markets frictions which cause long-run deviations from (constrained) efficient allocations. Indeed, by obtaining analytical solutions to the planner's problem, we show how coordinated monetary-fiscal policy that involves both fiscal tools can replicate constrained-efficient allocations.

Related Literature. First, our paper belongs to the recent literature on the transmission of uncertainty shocks in New Keynesian models.[2] Ilut and Schneider (2014) and Basu and Bundick (2017) study uncertainty-induced aggregate demand recessions in models where government bonds are in zero net supply. While these papers successfully capture the economic downturns resulting from high uncertainty, the household portfolio choice channel highlighted in our paper is missing. Christiano, Motto and Rostagno (2014) show that high uncertain can negatively affect credit supply through a financial accelerator mechanism. Leduc and Liu (2016) investigate the effects of uncertainty shocks from the perspectives of search frictions and employment relationships.

[1]This is sometimes labeled as "monetary stabilization" (e.g., in Bayer et al., 2019) since it is typically assumed that taxes adjust passively to accommodate interest rate changes so the government budget constraint is satisfied. However, it can be confusing because at the heart of such policies is the distortionary effects of taxes on household portfolios.

[2]In addition to the New Keynesian models, the literature on uncertainty shocks has investigated the "wait-and-see" effect (Bloom, 2009; Bloom et al., 2018) and various types of interactions between uncertainty and financial frictions (Gilchrist et al., 2014; Alfaro et al., 2018; Arellano et al., 2019; Elenev et al., 2021; Di Tella, 2017; Bianchi et al., 2018).

Fernández-Villaverde et al. (2015) study the impact of fiscal policy volatility in an environment with government bonds. However, since they study a representative-agent economy where Ricardian equivalence holds, the effect of non-distortionary fiscal policy studied in our paper is absent. Schaab (2020) studies the transmission of aggregate (macro) uncertainty shocks in a New Keynesian model with idiosyncratic labor market risk. In contrast, our paper focuses on shocks to idiosyncratic (micro) uncertainty.

More closely related to our paper is Bayer et al. (2019), where they examine the interplay between idiosyncratic income risk and household portfolio decisions in a quantitative HANK model. Broadly speaking, both papers share the common theme that uncertainty leads to portfolio rebalancing and aggregate fluctuations, but the types of risk and the mechanisms generating imperfect substitutability between capital and government bonds are different. Bayer et al. (2019) follows the Aiyagari (1994) tradition and study idiosyncratic *labor income risk* faced by the households, whereas in this paper, we focus on idiosyncratic *investment risk* as in Angeletos (2007) and Bloom (2009). In Bayer et al. (2019), capital stock is illiquid compared to bonds, so households hold for bonds for better *consumption smoothing*,[3] while in our paper, capital bears more idiosyncratic risk and households hold bonds as an *insurance*.[4] In this regard, our paper offers a new mechanism that also contributes to the empirical patterns established in their paper. In addition, we closely examine the roles of different policy tools in stabilizing aggregate demand and characterize the optimal monetary-fiscal policy, which are complementary to the distributional results in Bayer et al. (2019). Finally, our model features more tractability and provides a more transparent display of different mechanisms.

Moreover, we echo the extensive HANK literature that emphasizes the role of fiscal policy when Ricardian equivalence breaks down under incomplete markets. For example, Kaplan, Moll and Violante (2018), Acharya and Dogra (2020), and Caramp and Silva (2021) have investigated fiscal-monetary interactions in New Keynesian models with technology and monetary shocks. Our paper contributes to this literature by highlighting the importance of fiscal policy in environments with uncertainty shocks and

[3] Recent papers that jointly study the liquidity service of government papers and monetary transmissions include Del Negro et al. (2017), Kiyotaki and Moore (2019) and Szőke (2019).

[4] In a similar vein, recent works by Di Tella (2020) and Brunnermeier, Merkel and Sannikov (2022) model government papers as a store of value in uncertain environments.

portfolio reallocations.

Our paper also links with the literature on safe asset shortage and macroeconomic dynamics (e.g., Caballero and Farhi, 2018; Acharya and Dogra, 2022). Distinct from the earlier studies, our mechanism does not rely on a binding zero lower bound on the nominal rate. We show that the scarcity of safe assets matters in the presence of nominal rigidity and uncertainty shocks, even when the economy is away from safety and liquidity traps.

Our analysis of optimal policy connects to the burgeoning literature on the optimal design of monetary-fiscal policy in models with nominal rigidity and incomplete markets (e.g., Acharya, Challe and Dogra, 2020; Bhandari et al., 2021; Le Grand, Martin-Baillon and Ragot, 2021; Dávila and Schaab, 2022; McKay and Wolf, 2022). Notably, the existing literature mostly builds on one-asset models and ignores the portfolio choice between risky and safe assets,[5] whereas in this paper, we carefully study the implications of household portfolio adjustment. Our paper also focuses on different aspects of welfare calculation from the existing literature. Specifically, we draw our attention to the aggregate demand externality created by incomplete markets while abstracting away from wealth inequality and distributional efficiency.

Our model also relates to the literature on money as a store of value against idiosyncratic risk. In this literature, money has similar functionalities as nominal government bonds as in our paper. In particular, we share the same interest with Di Tella (2020) and Brunnermeier, Merkel and Sannikov (2022) in showing that the existence of nominally safe assets have important implications for monetary transmissions. In addition, we demonstrate how valuations of safe assets interact with nominal rigidity and how stabilization policies work differently with and without nominal government debts.

With regard to the methodology, we build on recent developments in continuous-time methods in macroeconomics and finance. Specifically, our formulation of household portfolio choice problem resembles those in Brunnermeier and Sannikov (2014, 2016), while our modeling of price adjustment frictions is along the same lines as Kaplan, Moll and Violante (2018).

[5]A notable expectation is Bilbiie and Ragot (2021), who study optimal monetary policy when households solve a portfolio problem between money, bond and capital. Unlike in our paper, the portfolio problem is motivated by household liquidity shocks and infrequent participation in financial markets.

Outline. Section 2 outlines the baseline model and our main numerical results. Section 3 discusses the transmission mechanisms of uncertainty shocks in our environment. Section 4 compares the baseline model with a model without safe government bonds. Section 5 studies the design of fiscal policy. Section 6 presents an extension with long-term government bonds. Section 7 concludes.

2 Model

In this section, we outline our baseline model. The model is based on the safe asset model of Brunnermeier, Merkel and Sannikov (2022) in which households solve a portfolio choice problem between idiosyncratically risky capital and safe nominal government bonds. The amount of idiosyncratic risk that agents face is time-varying. In times of high risk, households' demand for safe government bonds increases, which generates deflation pressures. We augment this model by introducing nominal rigidities in the form of price adjustment costs as in Rotemberg (1982). Nominal rigidities prevent the immediate revaluation of nominal safe assets, so that deflationary pressures from flight to safety create a demand shortage.

2.1 Setup

Overview. We study a continuous-time, infinite-horizon economy. The economy is populated by a continuum of households who own (risky) physical capital and (safe) government bonds. There exists another continuum of intermediate goods firms who rent capital from households to produce differentiated intermediate goods and sell them to a competitive sector of final goods firms. Final goods firms combine intermediate goods to a final good using a CES production technology.

The model features two key frictions: (1) incomplete markets limit idiosyncratic risk sharing by households and (2) intermediate goods firms face nominal price adjustment frictions.

We use the final good as a numeraire. That is, if the price level is P_t, one dollar is worth $1/P_t$ in terms of final goods.

Households and Assets. There is a continuum of households indexed by $i \in [0,1]$. All households have logarithmic preferences over consumption streams with identical time preference rate ρ,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i \mathrm{d}t\right]. \tag{1}$$

Each household manages physical capital that produces a flow of capital services $u_t^i k_t^i \mathrm{d}t$ which the household rents out to intermediate goods firms at rental price p_t^R (per unit of capital services). Here, k_t^i is the quantity of capital managed by household i at time t and u_t^i is the utilization rate chosen by the household. In addition to providing capital services, capital holdings may be subject to taxation by the government and they entitle households to profit distributions from intermediate goods firms to be specified below. Capital of household i evolves according to

$$\mathrm{d}k_t^i = \underbrace{\left[\Phi(\iota_t^i) - \delta(u_t^i)\right]}_{:=g(u_t^i,\iota_t^i)} k_t^i \mathrm{d}t + \tilde{\sigma}_t k_t^i \mathrm{d}\tilde{Z}_t^i + \mathrm{d}\Delta_t^{K,i}, \tag{2}$$

where $\mathrm{d}\Delta_t^{K,i}$ represents agent i's market transactions in physical capital, $\iota_t^i k_t^i \mathrm{d}t$ are the agent's physical investment expenditures (in final output goods), $\Phi(\cdot)$ is an increasing and concave function that captures adjustment costs in capital accumulation, $\delta(\cdot)$ is an increasing and convex function that makes capital depreciation dependent on utilization u_t^i, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across households i. $\tilde{Z}^i$ introduces agent-specific idiosyncratic risk. $\tilde{\sigma}_t$ is an exogenous process that governs the magnitude of idiosyncratic risk faced by agents. We assume that $\tilde{\sigma}_t$ is driven by some (aggregate) Brownian motion Z that is independent of all the idiosyncratic $\tilde{Z}^i$ processes.

The first key friction in the model is that households are unable to share idiosyncratic risk due to market incompleteness. While they are allowed to trade physical capital and government bonds, they cannot write financial contracts contingent on individual $\tilde{Z}^i$ histories. As a consequence, all households have to bear the idiosyncratic risk inherent in their physical capital holdings.

We formulate the decision problem of household i as a standard consumption-portfolio choice problem that does not make explicit reference to the capital trading process $\mathrm{d}\Delta_t^{K,i}$ as a choice variable and instead works with the shares of net worth invested in capital and bonds, respectively.[6] Let n_t^i be the net worth of household i and let θ_t^i be the share of net worth invested in bonds. Then household i's net worth evolves according to

$$\mathrm{d}n_t^i = n_t^i \left[\theta_t^i \mathrm{d}R_t^B + (1-\theta_t^i)\mathrm{d}R_t^{K,i}(u_t^i, \iota_t^i)\right] - c_t^i \mathrm{d}t, \tag{3}$$

where $\mathrm{d}R_t^B$ and $\mathrm{d}R_t^{K,i}(\cdot,\cdot)$ are the infinitesimal return processes for bonds and capital, respectively. We describe these returns in the following.

Let q_t^K denote the market price of capital and let $q_t^B := \frac{B_t}{P_t}\frac{1}{K_t}$ the real value of the total bond stock in the economy per unit of aggregate capital. Here, $K_t := \int_0^1 k_t^i \mathrm{d}i$ is the capital stock and B_t is the nominal face value of the outstanding stock of government bonds. Bonds make nominal interest payments to their holders at floating rate i_t set by the government. We assume that B_t evolves locally deterministically,

$$\mathrm{d}B_t = \mu_t^B B_t \mathrm{d}t,$$

for reasons that we discuss below when specifying government policy. We postulate that q_t^B and q_t^K follow a generic Ito evolution

$$\mathrm{d}q_t^B = \mu_t^{q,B} q_t^B \mathrm{d}t + \sigma_t^{q,B} q_t^B \mathrm{d}Z_t, \qquad \mathrm{d}q_t^K = \mu_t^{q,K} q_t^K \mathrm{d}t + \sigma_t^{q,K} q_t^K \mathrm{d}Z_t$$

with drifts $\mu_t^{q,B}$, $\mu_t^{q,K}$ and (aggregate) volatility loadings $\sigma_t^{q,B}$, $\sigma_t^{q,K}$ that are determined in equilibrium. The return on bonds is simply the interest income minus inflation

$$\begin{aligned}\mathrm{d}R_t^B &= i_t \mathrm{d}t + \frac{\mathrm{d}\left(1/P_t\right)}{1/P_t} \\ &= \left(i_t - \mu_t^B + g_t + \mu_t^{q,B}\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t \end{aligned} \tag{4}$$

where the second equality uses the identity $1/P_t = q_t^B K_t / B_t$ by definition of q_t^B.[7] Importantly, the return on bonds is not subject to idiosyncratic risk.

[6]The trading process $\mathrm{d}\Delta_t^{K,i}$ can be backed out ex post from the solution to this portfolio problem.

[7]We also use here $\mathrm{d}K_t = g_t K_t \mathrm{d}t$ with $g_t := \int_0^1 g(u_t^i, \iota_t^i) k_t^i \mathrm{d}i / K_t$, which follows readily from aggregating the individual capital evolutions (2).

The return on capital for household i is

$$\mathrm{d}R_t^{K,i}(u_t^i,\iota_t^i) = \overbrace{\left(\frac{p_t^R u_t^i}{q_t^K} - \frac{\iota_t^i}{q_t^K} + \frac{\omega_t}{q_t^K} - \tau_t^K\right)\mathrm{d}t}^{\text{dividend yield}} + \overbrace{\frac{\mathrm{d}\left(q_t^K\left(k_t^i - \Delta_t^{K,i}\right)\right)}{q_t^K\left(k_t^i - \Delta_t^{K,i}\right)}}^{\text{capital gain}}$$
$$= \left(\frac{p_t^R u_t^i - \iota_t^i}{q_t^K} + \frac{\omega_t}{q_t^K} - \tau_t^K + g(u_t^i,\iota_t^i) + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^i + \sigma_t^{q,K}\mathrm{d}Z_t. \quad (5)$$

The dividends per unit of capital consist of four parts: revenues from renting out capital to intermediate goods firms, $p_t^R u_t$, physical reinvestment in the capital stock, $-\iota_t$, profit transfers from intermediate goods firms, ω_t, and taxes levied by the government, $-\tau^K q_t$. The second term in the first line corresponds to the capital gains, which are the result of both changes in the capital price q_t^K and changes in the quantity of capital before market transactions $(k_t^i - \Delta_t^{K,i})$ due to investment, depreciation, and idiosyncratic shocks.

In total, household i's problem is to choose consumption $\{c_t\}_{t=0}^{\infty}$, capital utilization $\{u_t\}_{t=0}^{\infty}$, physical investment $\{\iota_t\}_{t=0}^{\infty}$, and the bond portfolio share $\{\theta_t\}_{t=0}^{\infty}$ in order to maximize utility (1) subject to the net worth evolution (3), where in the latter the return expressions are given by equations (4) and (5). When making decisions, the household takes initial net worth n_0^i and the processes $\{p_t^R, q_t^B, q_t^K, g_t, \omega_t, i_t, \tau_t^K, \mu_t^B\}$ as given.

Intermediate Goods Firms. There is a continuum of monopolistically competitive intermediate goods firms indexed by $j \in [0,1]$. Each intermediate goods firm operates a linear production technology

$$y_t^j = a\hat{k}_t^j$$

for a differentiated good of variety j. Here, a denotes the level of productivity and $\hat{k}_t^j$ are effective capital input services rented from households at (unit) rental price p_t^R.

Intermediate goods firms quote their output prices P_t^j in terms of the nominal unit. The second of the two key frictions in the model is that price adjustments are subject to adjustment costs as in Rotemberg (1982). Specifically, the nominal price P_t^j of firm j evolves according to

$$\mathrm{d}P_t^j = \pi_t^j P_t^j \mathrm{d}t$$

and the firm incurs a flow adjustment cost $\frac{\kappa}{2}\left(\pi_t^j\right)^2 Y_t \mathrm{d}t$ when choosing price inflation π_t^j, where $\kappa > 0$ is a parameter and Y_t is the aggregate output.

In total, the flow profit of firm j is given by

$$\varpi_t^j \mathrm{d}t := \left[\frac{P_t^j}{P_t} a\hat{k}_t^j - p_t^R \hat{k}_t^j - \frac{\kappa}{2}\left(\pi_t^j\right)^2 Y_t\right] \mathrm{d}t. \tag{6}$$

The firm chooses a path of price inflation π_t^j and capital inputs $\hat{k}_t^j$ to maximize the present value of flow profits

$$\mathbb{E}\left[\int_0^\infty \xi_t \varpi_t^j \mathrm{d}t\right]$$

subject to the demand for intermediate good variety j (equation (8) below). Future profits are discounted using the stochastic discount factor $\xi_t = e^{-\rho t}\frac{1}{C_t}$ where $C_t = \int_0^1 c_t^i \mathrm{d}i$ is the aggregate household consumption.[8]

We assume that the both the profit flow ϖ_t^j of intermediate goods firms and the price adjustment costs $\frac{\kappa}{2}\left(\pi_t^j\right)^2 Y_t$ are rebated to the household proportionally to their capital holdings.[9] The transfer received by households per unit of capital is

$$\omega_t = \frac{\int_0^1 \left[\varpi_t^j + \frac{\kappa}{2}\left(\pi_t^j\right)^2 Y_t\right] \mathrm{d}j}{K_t}. \tag{7}$$

We consider a symmetric equilibrium where the price paths are the same for all firms ($\pi_t^j = \pi_t$). In Appendix A, we present a detailed derivation of the New Keynesian Phillips Curve in this context.

Final Goods Firms. Final goods are produced by a competitive sector of firms using a CES technology

$$Y_t = \left[\int_0^1 \left(y_t^j\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}j\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

[8] While different from any individual household's SDF, the use of this SDF maximizes the value of the profit share claims for any agent. This is the case because we distribute profits proportional to capital holdings which are idiosyncratically risky. See Brunnermeier et al. (2022) for an in-depth discussion of this weighted-average SDF.

[9] This is consistent with the redistribution strategy according to factor input shares as suggested by Kaplan, Moll and Violante (2018). Note that we include price adjustment costs into the transfers for simplicity. At the expense of additional notation, we could also make price adjustment cost a resource cost without significantly altering any of our results.

where y_t^j is the input quantity of intermediate good variety j and ε is the elasticity of substitution across varieties. By standard CES cost minimization, final goods producers have downward-sloping demand curves

$$y_t^j = \left(\frac{P_t^j}{P_t}\right)^{-\varepsilon} Y_t \tag{8}$$

and the nominal price of the final good is given by

$$P_t = \left[\int_0^1 \left(P_t^j\right)^{1-\varepsilon} \mathrm{d}j\right]^{\frac{1}{1-\varepsilon}}.$$

Government. There is a government that issues floating-rate nominal government bonds and imposes taxes on capital. As stated previously, the total nominal quantity of outstanding bonds is denoted by B_t, its growth rate by μ_t^B, the nominal interest rate by i_t, and the proportional capital (value) tax rate by τ_t^K. The government's choices of the three policy variables, nominal interest i_t, the bond issuance rate μ_t^B, and the capital tax rate τ_t^K, must satisfy the flow budget constraint

$$i_t B_t = \mu_t^B B_t + P_t \tau_t^K \cdot q_t^K K_t. \tag{9}$$

That is, the total interest payments $i_t B_t$ have to funded by new bond issuance $\mu_t^B B_t$ or taxes $P_t \tau_t^K \cdot q_t^K K_t$.

The budget equation (9) makes apparent why we have assumed that bond growth $\mathrm{d}B_t/B_t$ is locally deterministic. In our framework, we have combined together all government liabilities in the single liability "bonds" B_t, so that no market transactions between different types of government liabilities (e.g. open market operations of exchanging bonds for reserves) can (discontinuously) change the quantity of outstanding bonds. Then, any $\mathrm{d}Z_t$-loading of $\mathrm{d}B_t/B_t$ would have to be absorbed by a $\mathrm{d}Z_t$-loading in tax revenues,[10] i.e. a shock-contingent tax/transfer scheme that continuously alternates between positive and negative transfer payments. Such contingent tax/transfer

[10] Interest payments cannot absorb a $\mathrm{d}Z_t$-loading of $\mathrm{d}B_t/B_t$ as long as the government issues standard nominal debt which must make locally deterministic interest payments.

schemes are highly counterfactual in our view, which motivates us to restrict attention to locally deterministic bond growth.[11]

Henceforth, we denote by

$$\breve{s}_t := \frac{P_t \tau_t^K q_t^K K_t}{B_t} \tag{10}$$

the ratio of the primary surplus to outstanding government debt. With this notation, the budget constraint (9) simplifies to

$$i_t = \mu_t^B + \breve{s}_t. \tag{11}$$

In what follows, we assume that the government sets the nominal rate i_t and the surplus-debt ratio $\breve{s}_t$ according to some policy rules

$$i_t = i\left(\mathbf{S}_t\right), \qquad \breve{s}_t = \breve{s}\left(\mathbf{S}_t\right) \tag{12}$$

and adjusts taxes τ_t^K and bond issuance μ_t^B to satisfy the budget constraint (11) and the definition equation of $\breve{s}_t$.[12] Here, $\mathbf{S}_t$ is a vector of aggregate variables (e.g., $\pi_t, \tilde{\sigma}_t$) chosen by the government. In our language, we will loosely identify i_t with "monetary policy" and $\breve{s}_t$ with "fiscal policy."

2.2 Equilibrium

Market Clearing. The goods market clears when the total output equals the sum of consumption and investment

$$\int_0^1 c_t^i \mathrm{d}i + \int_0^1 \iota_t^i k_t^i \mathrm{d}i = Y_t. \tag{13}$$

The bond market clearing condition states that bonds held by households must equal bonds supplied by the government

$$\int_0^1 \theta_t^i n_t^i \mathrm{d}i = \frac{B_t}{P_t}. \tag{14}$$

[11]It should be emphasized that our assumption does not rule out state-contingent changes in the capital tax *rate*. It merely rules out transfer schemes whose cash flows directly load on the Brownian innovations $\mathrm{d}Z_t$.

[12]In general, the assumption of automatically adjusting taxes may lead to equilibrium multiplicity. However, the model always features at most one equilibrium in which the value of government debt is positive and stationary. Compare Brunnermeier, Merkel and Sannikov (2020) for a discussion in a flexible price version of our model. In the following, we always select this equilibrium. This choice is in full analogy to the standard equilibrium selection in models of money as a medium of exchange.

Equilibrium Definition. Natural state variables in our model are $\tilde{\sigma}_t$, B_t, K_t, and P_t. It is possible to reduce the dimensionality of the state space from four to two dimensions. First, because of the AK nature of the economy, the capital stock K_t only matters for the scale of the economy. We can eliminate it from the state space by considering descaled quantities. Second, the states B_t and P_t only matter through their impact on the real value of government bonds, B_t/P_t. Thus, we can equivalently use the normalized value of the bond stock q_t^B as our second state variable. Recall its definition

$$q_t^B := \frac{B_t}{P_t K_t}.$$

The dynamics of q_t^B follow by Ito's lemma from the dynamics of bond supply B_t, the price level P_t, and aggregate capital K_t:

$$\mu_t^{q,B} = \mu_t^B - \pi_t - g_t, \quad \sigma_t^{q,B} = 0. \tag{15}$$

Importantly, the value of nominal bonds is directly linked to the price level. In the presence of price stickiness, q_t^B is also locally deterministic ($\sigma_t^{q,B} = 0$), i.e. it becomes a slow-moving state instead of the instantaneously adjusting equilibrium price that it would be under flexible nominal prices.

Definition 1 (Markovian equilibrium) *An equilibrium consists of laws of motion for the state variables $\{\tilde{\sigma}_t, q_t^B\}$ and of equilibrium mappings*

$$\left\{q^K(\tilde{\sigma}_t, q_t^B), \vartheta(\tilde{\sigma}_t, q_t^B), u(\tilde{\sigma}_t, q_t^B), \iota(\tilde{\sigma}_t, q_t^B), \pi(\tilde{\sigma}_t, q_t^B), i(\tilde{\sigma}_t, q_t^B), \check{s}(\tilde{\sigma}_t, q_t^B)\right\}$$

that satisfy

- *household and firm optimality,*
- *the policy rule (12),*[13]
- *market clearing conditions (13) and (14),*
- *the consistency requirement (15).*

[13]The government budget constraint (11) and the definition of $\check{s}_t$ can then be used to back out the tax rate τ_t^K and the bond growth rate μ_t^B.

2.3 Solution Method

We solve the household problem using a recursive approach as in Brunnermeier, Merkel and Sannikov (2022) and intermediate goods firms' price setting problems using the stochastic maximum principle. Details are outlined in Appendix B. We collect and discuss key model equations below in Section 3 where we explain the model mechanisms. We proceed to solve for equilibrium mappings

$$\left\{q^K(\tilde{\sigma}_t, q_t^B), \vartheta(\tilde{\sigma}_t, q_t^B), u(\tilde{\sigma}_t, q_t^B), \iota(\tilde{\sigma}_t, q_t^B), \pi(\tilde{\sigma}_t, q_t^B), i(\tilde{\sigma}_t, q_t^B), \check{s}(\tilde{\sigma}_t, q_t^B)\right\}$$

numerically using an iterative method as in Brunnermeier and Sannikov (2016). Descriptions of the numerical algorithm can be found in Appendix C.

2.4 Quantitative Results

In this section, we present a calibration of our model and illustrate its main quantitative predictions.

Calibration. First, we calibrate the time discount rate ρ to match the household portfolio share of safe assets of 15% in the stochastic steady state, which is roughly consistent the empirical evidence provided by Bayer et al. (2019).

We then assume the following functional forms for capital adjustment cost $\Phi(\cdot)$ and depreciation rate $\delta(\cdot)$. Following standard arguments, the investment function is concave while the depreciation rate function is convex,

$$\Phi(\iota_t) = \frac{1}{\phi}\log(1+\phi\iota_t), \quad \delta(u_t) = \frac{\bar{\delta}}{2}u_t^2.$$

We set $\phi = 5$ as in Di Tella and Hall (2022). The intercept of the depreciation rate function $\bar{\delta}$ is chosen to target a depreciation rate of around 7% in the stochastic steady state (e.g., Kaplan, Moll and Violante, 2018). For the New Keynesian block, we follow Kaplan, Moll and Violante (2018) in setting $\varepsilon = 10$ and $\kappa = 100$ to match the slope of the Phillips Curve to 0.1.

We adopt the Heston (1993) model of stochastic volatility for the law of motion of uncertainty $\tilde{\sigma}_t$. That is,

$$\mathrm{d}\tilde{\sigma}_t^2 = b(\bar{\sigma}^2 - \tilde{\sigma}_t)\mathrm{d}t - \nu\tilde{\sigma}_t\mathrm{d}Z_t.$$

The calibration of the uncertainty process follows from Merkel (2020). Specifically, we choose the value of $\{b, \bar{\sigma}^2, \nu\}$ to match some of the important moments of $\{\tilde{\sigma}_t\}$ to those of the series of establishment-level TFP shocks reported by Bloom et al. (2018).

For expositional purposes, we first consider an economy with constant policy variables i and $\breve{\mu}^B$.[14] We choose the fiscal policy to target a 1% primary surplus (% of GDP) in the stochastic steady state. The level of nominal rate i is chosen such that the steady-state inflation is zero. In Section 4, we present another parameterization where the nominal interest rate responds negatively to uncertainty shocks.

Table 1 summarizes the baseline calibration that will be used in the quantitative analysis.

Parameter	Description	Value	Target
Preference			
ρ	Discount Rate	0.07	$\vartheta^{SS} = 15\%$, Bayer et al. (2019)
Production			
a	Productivity	0.18	Standard
$\bar{\delta}$	Depreciation Rate Function	0.085	$\delta^{SS} \approx 7\%$
ϕ	Investment Adjustment Function	5	Di Tella and Hall (2022)
New Keynesian Block			
ε	Elasticity of Substitution	10	Kaplan et al. (2018)
κ	Menu Cost	100	Kaplan et al. (2018)
Uncertainty			
b	Speed of Adjustment	0.15	Bloom et al. (2018)
$\bar{\sigma}^2$	Mean	0.085	Bloom et al. (2018)
ν	Volatility	0.037	Bloom et al. (2018)
Monetary-Fiscal Policy			
i	Interest Rate	2.3%	$\pi^{SS} = 0$
$\breve{s}$	Fiscal Policy	0.8%	1% Primary Surplus

Table 1: Model Parameterization

Numerical Results. Consider an economy that is initially (at $t = 0$) in the stochastic steady state where $\tilde{\sigma}_0^2 = \bar{\sigma}^2$. We compute the impulse response functions (IRFs) to an uncertainty shock by comparing the results of the following two simulations

[14]Note that in the unique stationary monetary equilibrium (see footnote 12), the price level and inflation are determinate and the Taylor principle is not required for equilibrium selection.

Simulation 1. At $t = 0$, the economy is hit by a positive uncertainty shock,[15] and no other shocks are realized hereafter.

Simulation 2. No shocks are realized forever so the economy stays in the stochastic steady state.

The IRFs are defined as the difference between paths of relevant variables observed in these two simulations. Figure 1 visualizes such dynamics of the economy following a positive uncertainty shock. We briefly describe these dynamics here. An in-depth discussion of the mechanism is deferred to Section 3.

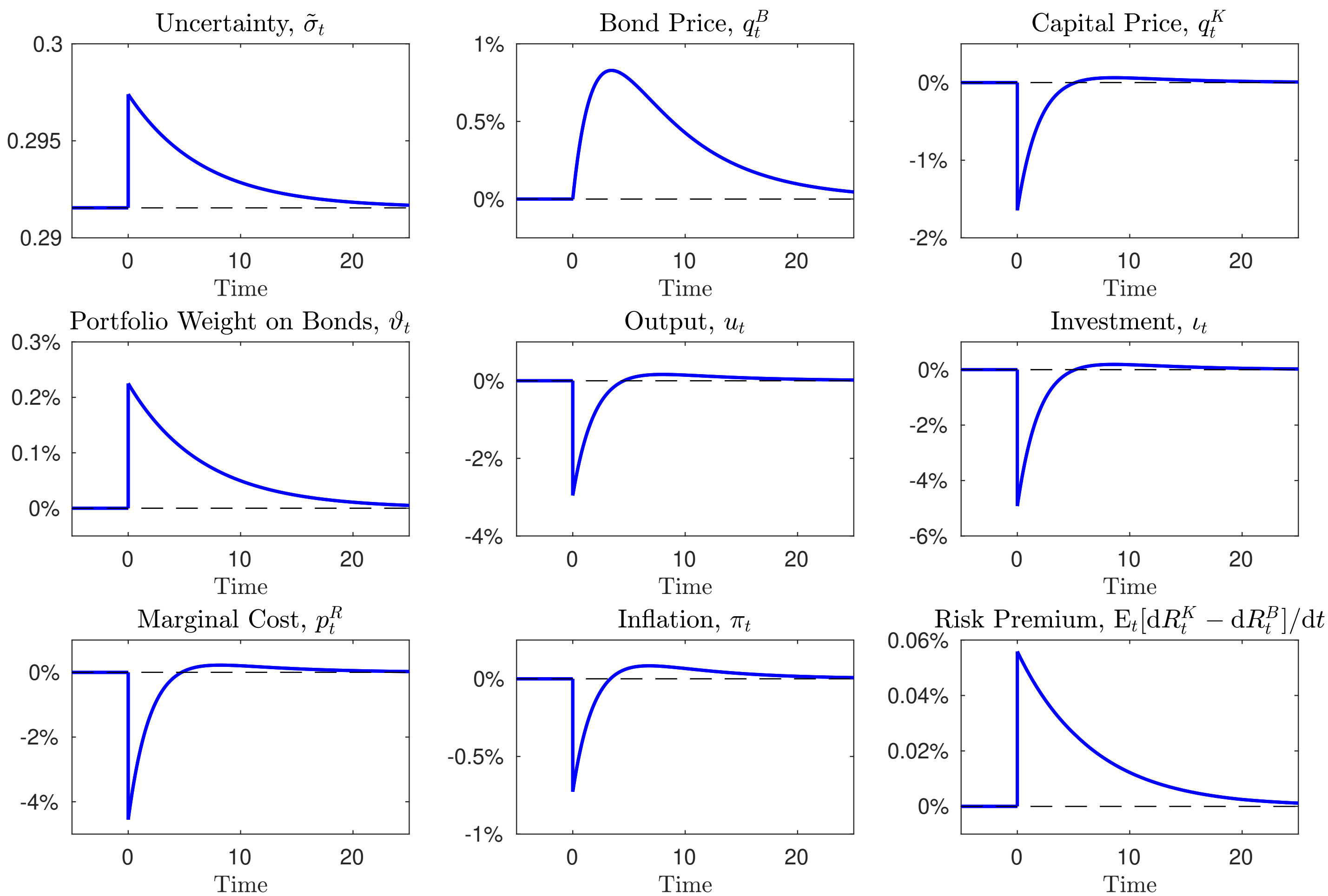

Figure 1: Impulse Response Functions (percentage differences)

[15]We calibrate the size of the shock to a generate a 3% recession in output.

We define the aggregate portfolio weight in bonds as the ratio of bond wealth to the total wealth of the economy,

$$\vartheta_t = \frac{B_t/P_t}{B_t/P_t + q_t^K K_t} = \frac{q_t^B}{q_t^B + q_t^K}. \tag{16}$$

As future uncertainty rises, the households increase their portfolio weight on bonds. When prices are sticky, the value of nominal bonds cannot adjust to clear the market, and the heightened demand for bonds leads to overshooting in the price of capital. As plotted in panel 2 and 3, the capital price q_t^K drops sharply when the shock hits, and then slowly converges to the new steady state level. In the meantime, the bond price q_t^B is sticky and rises gradually over time.

The drop in the price of capital in turn depresses effective demand. Both investment and aggregate output fall. The recession in aggregate demand reduces the marginal cost facing intermediary goods producers and increase their markups. Since the current price level is too high, the firms optimally set lower prices in the future, resulting in a period of disinflation. Finally, households demand a higher risk premium as the return on capital becomes more uncertain.

In summary, our model implies that uncertainty leads to recessions in consumption, investment, and output, disinflation, and portfolio rebalancing towards safe bonds. It is worth noting that our model predicts that the return on capital increases while the real economy is in a downturn following positive uncertainty shocks. Hence, our model can potentially account for the recent disconnect between the return on capital and the real economy (e.g., Caballero and Simsek, 2021).[16]

3 Mechanism

3.1 Solving the Household Problem

As detailed in Appendix B, our setup leads to an analytical solution to the household problem.

[16]Our current calibration does not fully match the magnitude of increases in risk premia following uncertainty shocks. To quantitatively capture this adjustment, the model needs a larger risk aversion. Nevertheless, we maintain the assumption of log utility in this paper for the sake of tractability.

Proposition 1 *The first-order conditions for household i are*

$$c_t^i: \qquad c_t^i = \rho n_t^i \tag{17}$$

$$\iota_t^i: \qquad \Phi'(\iota_t^i) = 1/q_t^K \tag{18}$$

$$u_t^i: \qquad \delta'(u_t^i) = p_t^R/q_t^K \tag{19}$$

$$\theta_t^i: \qquad \mathbb{E}_t\left[\mathrm{d}R_t^K - \mathrm{d}R_t^B\right]/\mathrm{d}t = (1-\theta_t^i)\left[(\sigma_t^{q,K})^2 + \tilde{\sigma}_t^2\right] \tag{20}$$

The first-order conditions have intuitive economic interpretations. With logarithmic utility, the households always consume a constant fraction ρ of their net worth. The investment rate is positively associated with the price of capital, which is a standard Tobin's q relationship. The utilization rate is increasing in the rental price of capital p_t^R, which represents the marginal revenue from production, and decreasing in the price of capital q_t^K, which represents the opportunity cost of depreciating more capital. Capital earns a risk premium on both aggregate risk and idiosyncratic risk, where the premium depends on the households' risk exposure $(1-\theta_t^i)$, the amount of aggregate risk $\sigma_t^{q,K}$, and the amount of idiosyncratic risk $\tilde{\sigma}_t$.

The first-order conditions (18) – (20) also imply that the household's decisions on investment ι_t, utilization u_t, and portfolio weight θ_t are independent of their own net worth or capital stock. Therefore, the households in this economy are *homogeneous up to scale*, as they all choose the same investment and utilization rates and hold the same portfolio regardless of their net worth.[17] While agents are heterogeneous with regard to their net worth n_t^i, the net worth distribution does not affect macro aggregates. It is in this sense that our analysis abstracts from distributional aspects that have been the focus of much of the HANK literature.

Note that the total net worth of the households consists of the total (real) value of government bonds and capital,

$$\int_0^1 n_t^i \mathrm{d}i = (q_t^B + q_t^K)K_t.$$

It is then useful to write the goods market clearing condition (13) as

$$\rho(q_t^B + q_t^K)K_t + \iota_t K_t = a u_t K_t. \tag{21}$$

[17]Henceforth, we omit the i superscript since all decision variables are identical across households.

Since all households choose the same portfolio $\theta_t^i = \theta_t$, the bond market clearing condition (14) can be simplified to

$$\theta_t = \frac{B_t/P_t}{B_t/P_t + q_t^K K_t} = \frac{q_t^B}{q_t^B + q_t^K}. \tag{22}$$

Comparing equations (16) and (22), we see that the individual portfolio weight must coincide with the aggregate portfolio weight, $\theta_t = \vartheta_t$.

3.2 Understanding Household Portfolio Decisions

To understand how household portfolio decisions respond to fluctuations in economic uncertainty, we derive a more intuitive condition that characterizes the dynamics of ϑ_t. Since the portfolio weight is uniform across households and thus not driven by any idiosyncratic shocks, we postulate that

$$\frac{\mathrm{d}\vartheta_t}{\vartheta_t} = \mu_t^{\vartheta}\mathrm{d}t + \sigma_t^{\vartheta}\mathrm{d}Z_t.$$

Plugging the expressions for asset returns (4), (5) and market clearing conditions (21), (22) into the household portfolio FOC (20), we obtain a closed form solution for μ_t^{ϑ}.

Proposition 2 (Bond Valuation Equation) *The law of motion of ϑ_t satisfies*

$$\mu_t^{\vartheta} = \rho - (1-\vartheta_t)^2\,\tilde{\sigma}_t^2 - \breve{s}_t. \tag{23}$$

In integral form,

$$\vartheta_t = \mathbb{E}_t\left\{\int_t^{\infty} e^{-\rho(s-t)}\vartheta_s\left[(1-\vartheta_s)^2\tilde{\sigma}_s^2 + \breve{s}_s\right]\mathrm{d}s\right\}. \tag{24}$$

This result clarifies that households' portfolio choice depends on only two processes, the path of future uncertainty $\{\tilde{\sigma}_s\}_{s\geq t}$ and the path of future surplus-debt ratios $\{\breve{s}_s\}_{s\geq t}$. Households demand more government bonds when they are valuable as insurance (higher $\tilde{\sigma}$) or when they are subsidized by government policy (higher $\breve{s}_t$).[18]

Importantly, household portfolio choice does *not* directly depend on output or capital utilization (u_t), inflation (π_t), the nominal interest rate (i_t), and any parameters governing the price setting behavior of firms (ε, κ). These variables and parameters only

[18] A higher surplus-debt ratio effectively subsidizes bond relative to capital holdings because it implies that the government taxes capital at a larger rate to make payouts to bond holders.

affect household portfolio choice between capital and bonds if (fiscal) policy adjusts $\breve{s}$ in response to them. This leads to the following separation result.

Proposition 3 (Separation of Portfolio Choice) *Suppose that the fiscal policy rule chooses $\breve{s}$ as a function of $(\tilde{\sigma}_t, \vartheta_t)$ only.*[19] *Then*

$$\vartheta_t = \vartheta(\tilde{\sigma}_t)$$

is a function of the exogenous uncertainty state $\tilde{\sigma}_t$ only and this function does not depend on the model parameters ε and κ.

In particular, portfolio choice is unaffected by inflation π_t, capital utilization u_t, nominal interest rates i_t and the degree of price stickiness.

This proposition has two important corollaries. First, monetary policy is *portfolio-neutral*: the nominal interest rate path i_t set by the monetary authority does not distort the portfolio choice between bonds and capital unless it induces changes in the fiscal policy instrument $\breve{s}_t$. Second, regardless of how sluggish nominal prices adjust, portfolio choice between capital and bonds is always "fast" in the sense that portfolios adjust precisely as under flexible prices. This second corollary plays an important role for the overshooting result discussed below.

Interest Rate Policy and Fiscal Policy. It is sometimes believed that interest rate policy can distort the households' demand for safe assets by changing the risk-free rate. We show that this is not necessarily the case when bonds are in a positive net supply. In our baseline model, the government only affects household portfolios through (capital) taxes in this economy. The reason is that the bond return (4) depends on the difference $i_t - \mu_t^B$ of nominal interest payments and bond issuance, which equals the surplus-debt ratio $\breve{s}_t$ by the government budget constraint (11) and is thus directly proportional to taxes.[20] Intuitively, changes in seigniorage income through bond issuance (μ_t^B) dilute the claim of existing bondholders in a way that exactly offsets changes in interest rates

[19]This includes, for example, simple policies such as a constant capital tax rate τ_t^K.

[20]The return on capital (5) also depends on government policy, but naturally only on the capital tax rate τ_t^K.

(i_t), unless they are associated with changes in the distribution of primary surpluses to bond holders (i.e. changes in $\breve{s}_t$).

To control household portfolio decisions, the government thus needs to adjust distortionary taxes. More specifically, the government can encourage investments in bonds by taxing capital and subsidizing bond holders through interest payments.

It should be emphasized that this strong result (i.e., interest rate policy alone is completely ineffective) is a special feature of our baseline model that is, for example, relaxed by the introduction of long-term bonds (Section 6). However, we would like to stress that fiscal policy is crucial in an environment with nominally safe assets, which is in sharp contrast to models without safe assets where fiscal policy only plays a minor role (see, for example, Section 4 and Basu and Bundick, 2017). We also highlight the possibility that seigniorage income of the government dampens the transmission of monetary policy, and that the response of fiscal policy becomes crucial when markets are incomplete.

3.3 The Transmission of Uncertainty Shocks

To understand how uncertainty affects output, we can rewrite the market clearing condition (21) as

$$u_t = \frac{1}{a}\left[\underbrace{\rho(q_t^K + q_t^B)}_{\substack{\text{consumption} \\ \text{demand}}} + \underbrace{\iota_t}_{\substack{\text{investment} \\ \text{demand}}}\right]. \tag{25}$$

The demand for final goods can be decomposed as demand for consumption and investment. The consumption demand is tied to the households' net worth through wealth effects. Net worth depends on the value of both bonds and capital. The investment demand ι_t instead depends on q_t^K through the Tobin's q relationship (18). To determine the short run output u_t, it is thus important to understand the behavior of the *levels of asset valuations*, q_t^K and q_t^B.

The solution of the household portfolio problem (equation (24)) determines *relative asset valuations*. Households invest more in nominal bonds as future uncertainty increases. Formally, the nominal wealth share ϑ_t rises. Because portfolio choice is separated from output determination by Proposition 3, we can treat ϑ_t in the following as an exogenous process and discuss how the components of aggregate demand depend on it.

Recall that

$$\vartheta_t = \frac{q_t^B}{q_t^B + q_t^K} = \frac{1}{1 + q_t^K / q_t^B}.$$

An increase in ϑ_t mechanically implies a drop in the price of capital relative to the value of bonds. However, the absolute movements in q_t^K and q_t^B hinge on the presence of price stickiness.

Equilibrium under Flexible Prices. When prices are flexible, intermediate goods firms' problem collapses to a static profit-maximization problem at each time and firms engage in constant markup pricing (see Appendix D for details). Price setting merely determines the capital rental price,

$$p_t^{R,flex} = a \frac{\varepsilon - 1}{\varepsilon}, \tag{26}$$

while the nominal price level P_t is free to adjust in order to clear the goods market, i.e. make equation (25) hold. In that equation, P_t enters through the scaled bond value $q_t^B = B_t / (P_t K_t)$, which it is inversely proportional to.

Together with equations (19), (22), and (26), we can solve the goods market clearing equation (25) for prices q_t^K and q_t^B analytically. For tractability, we keep using the following functional forms: $\Phi(\iota) = \log(1 + \phi\iota)/\phi$ and $\delta(u) = (\bar{\delta}/2)u^2$, which then imply that under flexible prices,

$$q_t^{K,flex} = \left[\frac{a^2}{\rho \bar{\delta}} \frac{\varepsilon - 1}{\varepsilon} \right]^{1/2} (1 - \vartheta_t)^{1/2}, \tag{27}$$

$$q_t^{B,flex} = \left[\frac{a^2}{\rho \bar{\delta}} \frac{\varepsilon - 1}{\varepsilon} \right]^{1/2} \frac{\vartheta_t}{(1 - \vartheta_t)^{1/2}}. \tag{28}$$

As uncertainty increases ($\vartheta_t \uparrow$), government bonds gain in value ($q_t^B \uparrow$) while capital loses ($q_t^K \downarrow$). Since $q_t^B \propto 1/P_t$, the increase in q_t^B implies a drop in the price level P_t.

Economically, an increase in ϑ_t induced by heightened uncertainty means that agents desire more bond savings and less capital savings. As a result, both the capital price

q_t^K and the physical investment rate ι_t fall, implying lower consumption demand due to capital wealth and investment demand on the right-hand side of equation (25). For fixed utilization u_t and bond values q_t^B, aggregate demand thus falls short of supply.

However, unlike capital, bond holdings cannot be used as an aggregate savings device. As households collectively demand more nominal bonds and less goods, the price level falls and the real value of bonds q_t^B appreciates. A wealth effect from increased bond wealth generates additional consumption demand that eliminates the shortfall in aggregate demand without any reduction in supply.[21]

Equilibrium under Sticky Prices: Impact Effect. When the price level P_t is sticky, it can no longer drop immediately in response to an uncertainty shock. There is no (immediate) deflationary revaluation of nominal wealth that raises consumption demand as under flexible prices. Instead, supply must adjusts to the shortfall in demand to clear the goods market. In this model, this adjustment happens through a reduction in capital utilization u_t.

In fact, in our continuous-time model, the state variable q_t^B for (scaled) bond wealth is perfectly rigid over short time periods and thus cannot adjust at all on shock impact.[22] As a consequence, when higher uncertainty raises ϑ_t, both total wealth $q_t^K + q_t^B = q_t^B / \vartheta_t$ and capital valuations $q_t^K = (1 - \vartheta_t)/\vartheta_t \cdot q_t^B$ fall. The former implies a reduction in consumption demand, the latter a reduction in investment demand through the Tobin's q relationship (18). Importantly, both components of aggregate demand in equation (25) are fully predetermined by the (separate) portfolio choice ϑ_t and the (rigid) state variable q_t^B. The only free variable to adjust on impact is u_t and a recession is inevitable.

In fact, the investment demand and the capital price always fall by more than under flexible prices. The reason is that when q_t^B is unable to adjust, a higher portfolio weight ϑ_t can only be achieved through a more substantial drop in the capital price q_t^K. That is, since the value of government bonds cannot adjust quickly to match fluctuations in the demand for bonds, uncertainty shocks depress the demand for capital even further.

[21] In fact, in this model aggregate supply even expands under flexible prices because the lower capital price q_t^K reduces the opportunity cost of higher capital utilization u_t.

[22] Formally, this is apparent in $\sigma_t^{q,B} = 0$ in equation (15).

Suppose the economy is in the steady state before an uncertainty shock hits. Immediately after uncertainty increases, the price of bonds is stuck at the pre-shock level q^B_{t-}, which is smaller than the post-shock flexible price value $q^{B,flex}_t$. At the onset of the shock, the price of capital is determined by the new portfolio weight ϑ_t, so

$$q^K_t = q^B_{t-}\frac{1-\vartheta_t}{\vartheta_t} < q^{B,flex}_t\frac{1-\vartheta_t}{\vartheta_t} = q^{K,flex}_t. \tag{29}$$

Thus, the price of capital *overshoots* relative to the flexible price response following uncertainty shocks when nominal prices are sticky.

This result is reminiscent of the classic overshooting model by Dornbusch (1976), who shows that stickiness in domestic prices results in overshooting in exchange rates, whereas in our model, the effective stickiness in bond valuations leads to overshooting in the price of capital.

Equilibrium under Sticky Prices: Adjustment Dynamics. After the shock, the price level P_t slowly decreases over time to bring it closer to its natural (flexible price) level, leading to a period of disinflation. Adjustment dynamics are guided by the Phillips curve resulting from intermediate goods firms' forward-looking price setting behavior and by the state equation (15). Under the simplifying assumption that agents do not expect any additional future shocks going forward, these two equations can be written as[23]

$$\mathrm{d}\pi_t = \left[\rho\pi_t - \frac{\varepsilon}{\kappa}\left(p^R_t - p^{R,flex}\right)\right]\mathrm{d}t \tag{30}$$

$$\mathrm{d}q^B_t = \left(i_t - \pi_t - \check{s}_t - g_t\right)\mathrm{d}t \tag{31}$$

where p^R_t and g_t are functions of utilization u_t, and the latter itself is fully determined by the demand equation (25) and the current bond value q^B_t.

The Phillips curve (30) is a forward-looking equation that encodes the standard price setting logic in New Keynesian models: firms' price setting is trading off a desire to minimize deviations of marginal unit costs p^R_t from their flexible price value $p^{R,flex}$ to achieve the statically optimal markup (second term in equation (30)) and a

[23]The fully stochastic Phillips curve that guides the actual model dynamics can be found in Appendix A. The equations stated here facilitate interpretation as they are significantly simpler.

desire to smooth inflation in order to minimize price adjustment costs (first term in equation (30)). After an increase in uncertainty, a lack of demand for capital inputs drives the capital rental price p_t^R below its flexible price value, so that

$$-\frac{\varepsilon}{\kappa}\left(p_t^R - p^{R,flex}\right) > 0$$

and equation (30) implies gradually increasing inflation. As this is a (forward-looking) backward equation, the gradual increase in inflation is achieved by an immediate reduction of π_t into deflationary territory.

Low and potentially negative π_t increases the rate of appreciation of the state variable q_t^B in equation (31), so that q_t^B gradually approaches its natural level $q_t^{B,flex}$ over time. As q_t^B becomes closer to $q_t^{B,flex}$, the marginal cost gap $p_t^R - p^{R,flex}$ in the Phillips curve (30) starts to shrink, thereby dampening further adjustments in inflation.[24]

It is in the state equation (31) that nominal interest rates i_t enter. By raising the nominal rate, the government expands nominal bonds at a faster rate μ_t^B (for given $\check{s}_t$), so that the price level has to adjust by less in order to expand the real quantity of safe assets q_t^B. Importantly, interest rate policy only acts on a state evolution (forward equation) not on some forward-looking choice condition (backward equation). As a consequence, interest rate policy can affect the speed of adjustment dynamics but not prevent the initial reduction in aggregate demand on shock impact.

3.4 Relations to the Existing Literature

Monetary Models with Idiosyncratic Risk. Recent works by Di Tella (2020) and Brunnermeier, Merkel and Sannikov (2020, 2022) have emphasized the role of nominal assets (e.g., money and government bonds) as an insurance against idiosyncratic risk. Our paper shares the same spirit that uncertainty shocks affect the precautionary savings of households, which leads to portfolio rebalancing between safe assets and risky assets ("flight-to-safety"). Further, our paper extends this literature by introducing nominal rigidity and highlighting the interaction between sticky prices and portfolio decisions.

[24]Whether q_t^B asymptotically reaches $q_t^{B,flex}$ depends on the long-run behavior of nominal rates as the Phillips curve is only consistent with a zero long-run output gap if inflation settles down at zero.

Safe Asset Shortage at the Zero Lower Bound (ZLB). Recently, a number of works (e.g., Caballero and Farhi, 2018; Acharya and Dogra, 2022) have examined the implications of safe asset scarcity. However, in this class of models, government policy affects the safety demand only through the nominal interest rate, and safe asset supply only matters for equilibrium outcomes when the nominal rate is constrained by the ZLB. In contrast, we show that when safe assets take the form of nominal government bonds whose value is slow to adjust due to nominal rigidity, it is important to consider the entire government budget constraint. In particular, safe asset supply can have a direct impact on macroeconomic dynamics even if the nominal rate is positive, and it is crucial to consider the response of fiscal policy to portfolio reallocations.

Government Bonds as a Liquid Asset. A recent strand of the New Keynesian literature focuses on the liquidity service of government debts. A notable example is Bayer et al. (2019), where households are subject to idiosyncratic income risk as in Aiyagari (1994). In their model, the capital stock is subject to transaction costs whereas bonds can be easily liquidated.[25] When uncertainty increases, the households hold more government bonds to facilitate future consumption smoothing. In this paper, we adopt an alternative framework and motivate the agents' portfolio choice problem with idiosyncratic investment risk (e.g., Angeletos, 2007; Bloom, 2009). The households instead face a risk-return trade-off between capital and bonds. When uncertainty rises, the households shed away from risk due to precautionary savings motives.

Despite having different foundations for household portfolio decisions, the overshooting mechanism emphasized here also applies to models where assets differ in their degrees of liquidity. In fact, the interaction between nominal rigidity and the valuation of nominal government bonds is ubiquitous in models with multiple assets. However, while the existing literature has quantitatively established that uncertainty shocks can trigger aggregate demand recessions, the role of nominal government bonds has not received adequate attention. In this regard, our paper further clarifies the transmission mechanisms behind previous quantitative results.

[25] Del Negro et al. (2017) provides a different modeling approach where government bonds ease the financial frictions facing entrepreneurs.

4 Comparison to Models without a Safe Asset

Previous literature has studied the impact of uncertainty shocks in New Keynesian models where government bonds are in zero net supply (e.g., Basu and Bundick, 2017). While abstracting away from portfolio reallocations following uncertainty shocks, the authors still find similar business cycle co-movements following an uncertainty shock. In this section, we contrast models without safe bonds to our baseline model and further highlight the importance of portfolio adjustment in shaping macroeconomic dynamics following uncertainty shocks.

The model without a safe asset corresponds to a special case of our model where the supply of bonds is zero ($B_t = 0$). In this case, the households are forced to choose a portfolio entirely in capital ($\vartheta_t = 0$).

The Source of Fluctuations. When $B_t = 0$, the short-term output is given by

$$u_t = \frac{1}{a}\Bigg[\underbrace{\rho q_t^K}_{\substack{\text{consumption}\\ \text{demand}}} + \underbrace{\iota_t}_{\substack{\text{investment}\\ \text{demand}}} \Bigg]. \tag{32}$$

Since household portfolios stay constant over time, the interaction between portfolio rebalancing and price stickiness is shut off. Instead, the recessions in consumption, investment, and output stem from inadequate responses of monetary policy. To further clarify the transmission mechanism of uncertainty shocks in this model, it is useful to revisit the household portfolio choice condition (20), which now becomes

$$\mathbb{E}_t\left[\mathrm{d}R_t^K\right]/\mathrm{d}t - r_t^f = (\sigma_t^{q,K})^2 + \tilde{\sigma}_t^2. \tag{33}$$

When uncertainty increases, the households demand a higher risk premium on capital. In this case, the risk free rate is given by the Fisher equation

$$r_t^f = i_t - \pi_t$$

and therefore directly controlled by the monetary authority through the policy rate.If monetary policy failed to lower the risk-free rate aggressively enough, the heightened risk premium requires a higher return on capital, which leads to a drop in capital price q_t^K. From equation (32), we see that a drop in q_t^K translates to recessions in consumption, investment, and output when prices are sticky.

Interest Rate Policy. Furthermore, Basu and Bundick (2017) show that first best allocations can be implemented in their environment if the nominal interest rate pegs the natural rate perfectly, in which case investment and output increase after uncertainty rises. Here, we prove a similar result but within our model. Plugging the return on capital (5) into equation (33), we see that

$$\frac{u_t a - \iota_t}{q_t^K} + g_t + \left[\mu_t^{q,K} - (\sigma_t^{q,K})^2\right] = r_t^f + \tau_t^K + \tilde{\sigma}_t^2. \tag{34}$$

In equilibrium, all variables on the left-hand side depend on q_t^K. So by changing the real rate r_t^f, monetary policy directly influences the required return on capital and therefore the price of capital q_t^K. Formally, interest rate policy alone is adequate in stabilizing the economy when government bonds are in zero net supply, if the nominal rate is perfectly aligned with the natural rate.

Proposition 4 *If bonds are in zero net supply, the flexible price equilibrium with zero inflation can be implemented with the following policy*

$$i_t = i_t^n, \quad \tau_t^K = 0.$$

The natural interest rate i_t^n is defined as

$$i_t^n = \rho + g_t^n - \tilde{\sigma}_t^2,$$

where g_t^n is the steady state growth rate of the aggregate capital stock (see Appendix D).

In contrast, when government bonds are in positive net supply, equation (29) shows that interest rates only affect the initial recession in q_t^K if they affect household portfolios. However, from the bond evaluation equation (23), we see that the household portfolio and the risk premium on capital depends only on the level of uncertainty $\tilde{\sigma}_t$ and $\breve{s}_t$, the surplus-debt ratio, which is only affected by the tax rate τ_t^K (and the relative price ϑ_t). When the tax rate is unresponsive to shocks, the effects of interest rate changes are offset by movements in bond growth. As have been summarized in Proposition 3, interest rate policy can only affect the portfolio of the households if it induces responses in fiscal policy (distortionary taxes).

Amplification. More importantly, even under the same monetary policy, the effects of uncertainty shocks are significantly amplified in the presence of nominal bonds. To test this prediction quantitatively, we numerically solve the model with $B_t = 0$, and compare the implied impulse response functions with the baseline model with $B_t > 0$. We specify a Taylor rule for the model without safe bonds

$$i_t = i^* + \varphi(\pi_t - \pi^*).$$

To ensure that monetary policy responses do not drive the results of comparison, we feed the same paths of $\{i_t\}$ into the baseline model when computing the impulse response functions.[26] Further, the government's interest payments become zero when $B_t = 0$. To maintain comparability, we also impose $\check{s}_t = 0$ on the baseline model in this exercise.

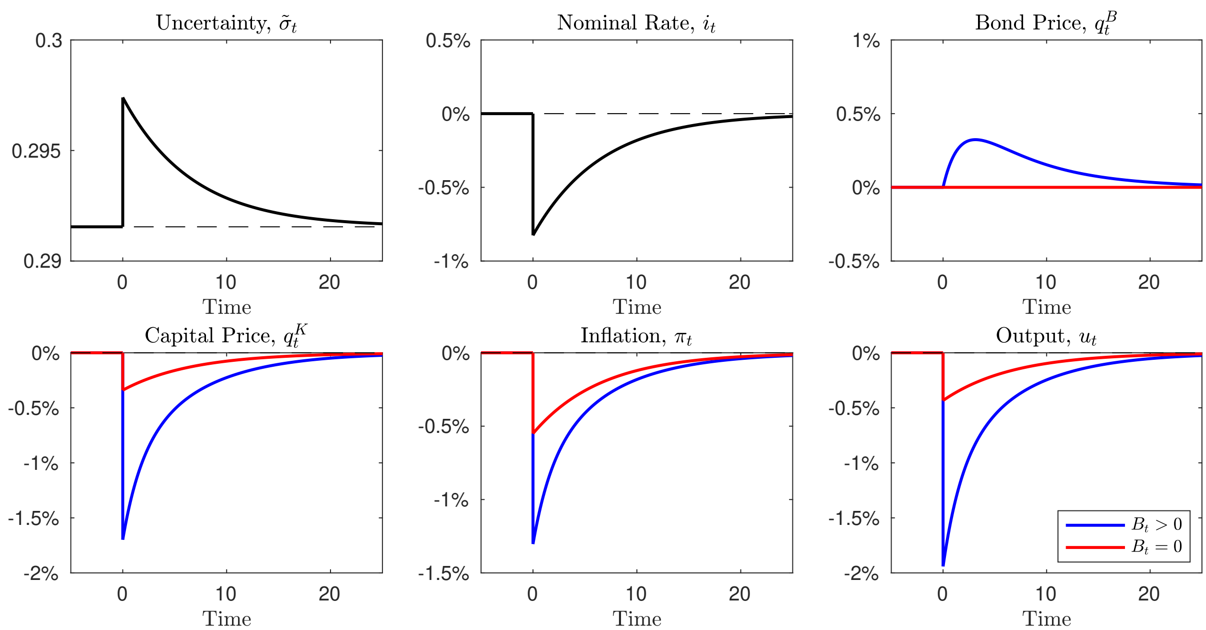

Figure 2: Impulse Response Function Comparison

Figure 2 compares the impulse response functions of the two models following the same uncertainty shocks and nominal rate responses. As shown in the figure, the model with nominal bonds generate deeper recessions in capital price and output, as well as

[26]Technically, we construct a mapping of uncertainty to the nominal rate $i_t = i(\tilde{\sigma}_t)$ and replace equation (12) with this mapping.

greater deflationary pressure. We then conclude that the interaction between nominal rigidity and household portfolio adjustment constitutes a powerful amplification mechanism.

5 Fiscal Policy

In previous sections, we have demonstrated how nominal rigidity leads to amplified recessions in the presence of nominal government bonds and how interest rate policy alone is inadequate for price stabilization. In this section, we examine how appropriately designed monetary and fiscal policy can stabilize an economy with uncertainty shocks and even implement constrained efficient allocations.

5.1 Non-distortionary Fiscal Policy

First, we show how fiscal policy can be utilized to stabilize uncertainty-induced fluctuations. Distortionary fiscal policy (e.g., taxes on capital) has very powerful effects in our economy as it changes both the short-term dynamics and the steady state. This section instead focuses on non-distortionary fiscal policies. Discussions on the design of distortionary taxes will be deferred to Section 5.2.

Lump-sum Taxes. Assume that in addition to the capital tax, each household also has to pay $\tau_t K_t \mathrm{d}t$ units of final goods as taxes over the time interval $[t, t + \mathrm{d}t]$. Since the mass of households is one, the aggregate flow of real lump-sum taxes is also $\tau_t K_t \mathrm{d}t$. The government can also make transfers to the households by selecting a $\tau_t < 0$. Further, we assume that the market for aggregate risk is complete.

Assumption 1 *The households can write contracts on aggregate variables.*

In this case, the households trade claims on future lump-sum taxes with each other. Importantly, the amount of capital tax paid by an individual household depends on its capital stock, which is subject to an idiosyncratic risk. Therefore, unlike the lump-sun tax, the capital tax is not tradable.

The government is long in the tax claims while the households are collectively short. Let $q_t^\tau K_t$ be the value of all tax claims at time t and $\mathrm{d}q_t^\tau = \mu_t^{q,\tau} q_t^\tau \mathrm{d}t$.[27] The return on the tax claims is

$$\mathrm{d}R_t^\tau = \frac{\tau_t}{q_t^\tau}\mathrm{d}t + \frac{\mathrm{d}(q_t^\tau K_t)}{q_t^\tau K_t} = \left(\frac{\tau_t}{q_t^\tau} + g_t + \mu_t^{q,\tau}\right)\mathrm{d}t.$$

In equilibrium, the price of tax claims q_t^τ is pinned down by a no-arbitrage condition between bonds and tax claims, and is ultimately a function of *future* tax rates,

$$q_t^\tau = q^\tau(\{\tau_s\}_{s=t}^\infty).$$

Essentially, the government expands the set of safe assets by introducing a tradable lump-sum tax. The total supply of idiosyncratic-risk-free assets to the household sector is now $q_t^G K_t = (q_t^B - q_t^\tau)K_t$. The dynamics of the economy is the same as before but with a new safe asset (i.e., with bonds $q_t^B K_t$ replaced by total government liabilities $q_t^G K_t$).[28] Instead of choosing a portfolio between capital and bonds, the households now allocate their portfolio between capital and all government liabilities. Importantly, the price of all safe assets q_t^G is no longer sticky, and the government can potentially stabilize the economy by choosing future tax rates $\{\tau_s\}_{s=t}^\infty$ such that q_t^G is at its flexible-price level.

Proposition 5 *Suppose that the rate of distortionary tax is a function of $(\tilde{\sigma}_t, \vartheta_t)$ only. For any such tax schemes, the associated zero-inflation flexible price equilibrium can be implemented if the nominal rate i_t and lump-sum taxes $\{\tau_s\}_{s=t}^\infty$ satisfy*

$$i_t = g_t^n, \quad q_t^{B,flex} = q_t^B - q^\tau(\{\tau_s\}_{s=t}^\infty).$$

The price of tax claims $q^\tau(\{\tau_s\}_{s=t}^\infty)$ is increasing in future tax rates. Its functional form, together with the price of safe assets under flexible prices $q_t^{B,flex}$ and the formula for the steady-state capital growth rate g_t^n, is derived in Appendices D and F.

As discussed in Section 3, the steady-state price of safe assets increases after a positive uncertainty shock. Without lump-sum taxes, the appreciation of safe bonds is achieved over time through disinflation. With lump-sum taxes and a complete market for aggregate risk, the government can rapidly boost the total supply of safe assets by committing to lower taxes (or higher transfers) in the future.

[27] We consider a case where the future path of lump-sum taxes $\{\tau_t\}$ is deterministic, so the value of tax claims does not bear any aggregate risk.

[28] See Appendix F for a formal discussion.

5.2 Optimal Capital Tax

Welfare and Constrained Efficiency. To study optimal monetary-fiscal policy in this economy, we first derive the value function of the households. The consumption of an individual household can be written as

$$c_t = \underbrace{(au_t - \iota_t)K_t}_{\text{aggregate consumption}} \times \underbrace{\frac{n_t}{N_t}}_{\text{wealth share}}, \tag{35}$$

where $N_t = (q_t^B + q_t^K)K_t$ is the total net worth of the economy. Since the households are homogeneous, the wealth share of any individual household is only subject to the idiosyncratic risk in its capital accumulation process. Specifically,

$$\frac{\mathrm{d}(n_t/N_t)}{n_t/N_t} = \frac{\mathrm{d}(k_t/K_t)}{k_t/K_t} = (1-\vartheta_t)\tilde{\sigma}_t \mathrm{d}\tilde{Z}_t.$$

Then the household value function can be obtained by taking log on both sides of equation (35) and integrate over time.

Proposition 6 *The households' value function can be written as*

$$\begin{aligned} V_0 &= \mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log c_t \mathrm{d}t\right] \\ &= \underbrace{\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log(au_t - \iota_t)\mathrm{d}t\right]}_{\text{“marginal productivity”}} + \underbrace{\mathbb{E}_0\left[\frac{1}{\rho}\int_0^\infty e^{-\rho t}\left[g(u_t,\iota_t) - \frac{(1-\vartheta_t)^2\tilde{\sigma}_t^2}{2}\right]\mathrm{d}t\right]}_{\text{growth-risk trade-off}} + \textit{constant}. \end{aligned}$$

The household's welfare depends first on a "marginal productivity" term which determines the fraction of aggregate capital stock that is transformed to consumption goods. The household also faces a growth-risk trade-off. While holding a portfolio more concentrated in safe assets reduces the household's exposure to idiosyncratic risk, it also lowers the equilibrium price of capital and investment (see equation (27)).

The constrained-optimal allocation is defined as the solution to the following problem where a planner maximizes the household value function subject to the market clearing constraints and the Tobin's q equation

$$\max_{u_t,\iota_t,\vartheta_t} V_0 \quad \text{s.t.} \quad au_t - \iota_t = \rho(q_t^B + q_t^K), \quad \vartheta_t = \frac{q_t^B}{q_t^K + q_t^B}, \tag{36}$$
$$\Phi'(\iota_t) = 1/q_t^K.$$

It is assumed that the government cannot directly distort the households' investment decision, so the Tobin's q equation appears as a constraint. It is apparent that the planner's problem collapses into a static problem at each time.

As is common in models with incomplete markets, the competitive equilibrium in our model features *pecuniary externality*. Specifically, the households fail to fully internalize the impacts of their decisions on aggregate prices such as q_t^K. Denote the solution to the planner's problem by $\iota_t^o, u_t^o, \vartheta_t^o$. With the parameter values in Table 1, the competitive equilibrium yields in steady state

$$\iota_t > \iota_t^o, \qquad u_t > u_t^o, \qquad \vartheta_t < \vartheta_t^o.$$

That is, the competitive equilibrium exhibits insufficient risk sharing and excessive investment and production and is therefore is constrained-inefficient.

Replicating the Optimal Portfolio Weight. In this section, we show that appropriately designed fiscal policy can replicate the constrained-efficient portfolio weight ϑ_t^o. The procedure takes three steps. First, solve for ϑ_t^o from the first-order conditions of the planner's problem. Then, obtain its drift $(\mu_t^\vartheta)^o$ by applying Ito's Lemma. Third, plug $(\mu_t^\vartheta)^o$ into the bond evaluation equation (23) to get the appropriate surplus-debt ratio $\breve{s}_t$ and tax rate on capital.

Figure 3 plots ϑ_t^o and the optimal τ_t^K as a function of uncertainty $\tilde{\sigma}_t$. The constrained-efficient portfolio puts more weight on bonds when uncertainty is high, due to stronger precautionary motives of the households. Meanwhile, the optimal tax rate is also increasing with uncertainty. When uncertainty is low, the planner subsidizes capital to encourage risk-taking, while when uncertainty is high, the planner taxes capital to achieve better insurance against idiosyncratic risk.

Apart from the pecuniary externality in optimal choice, the economy also sustains welfare loss from monopolistic competition among the intermediary goods firms. To tackle the market power of intermediary goods producers, the government needs additional tools, where the New Keynesian literature typically assumes a subsidy on the intermediary goods producers (Woodford, 2003; Galí, 2008). The same type of output subsidy also works in our model. We relegate the details to Appendix E.

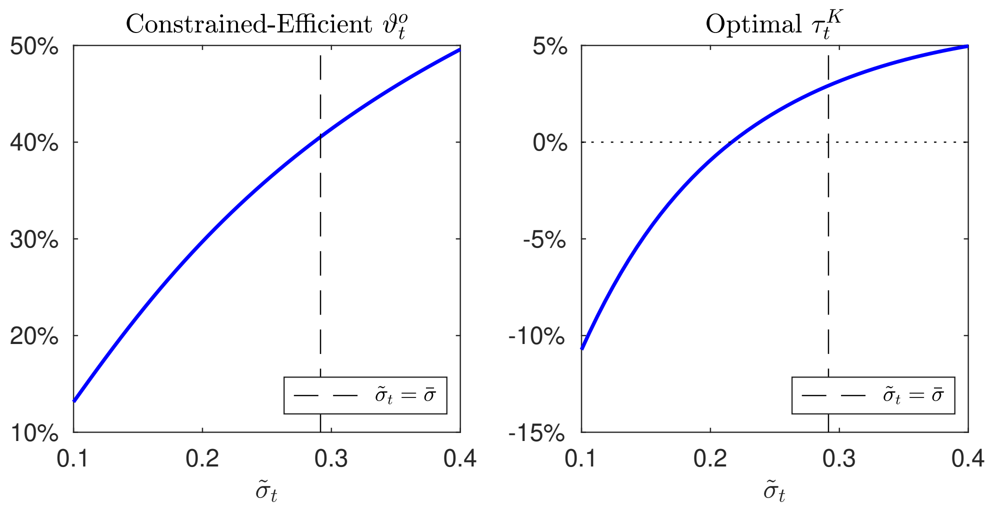

Figure 3: Optimal capital tax under parameterization in Table 1

To restore constrained efficiency, the government can take the following two steps. First, it can eliminate the distortions associated with nominal rigidity using a lump-sun tax as described in Proposition 5. Second, it can choose distortionary taxes (and subsidies on intermediary goods firms) optimally so the flexible-price equilibrium is constrained efficient.

6 Long-term Bonds

So far, we have assumed that the government issues nominal bonds with infinitesimally short duration. As infinitesimally short-term nominal bonds are essentially the same thing as money, this setting requires the nominal price of bonds to be always 1. As a consequence, when the price price level is sticky, then so must be the real value of government bonds – and thus safe assets – in the economy.

In this section, we generalize our model to allow for long-duration government bonds. Then the tight link between the price level and the market value of safe assets breaks down, yet our conclusions are not radically altered:

First, we show that even when the government issues long-term debt, the determinants of portfolio choice and the nominal wealth share ϑ_t remain essentially unchanged. As a consequence, nominal interest rate policy is still unable to affect portfolio choice directly (as opposed to indirectly through its fiscal consequences). Conditional on the safe asset value q_t^B, aggregate demand remains determined independently of nominal variables.

Second, with long-term bonds there is a relative price between bonds (safe assets) and money that can adjust instantaneously even under sticky prices. The adjustment is tightly linked to the path of future policy rates by a condition that largely resembles the expectations hypothesis. This enlarges the power of monetary policy to stabilize aggregate demand on shock impact, as it can now move the safe asset value q_t^B instantaneously by adjusting the path of future policy rates. However, with the single interest rate policy instrument, monetary policy is generally unable to simultaneously close the gap between safe asset demand and supply on impact of an uncertainty shock and keep the nominal rate close to the natural rate as to not introduce future distortions in price setting.

The setup of our generalized model is as before, except that we replace the assumptions on government bonds: instead of floating-rate bonds, the government issues bonds with a geometric maturity structure with duration $1/\lambda$ where $\lambda > 0$ is a model parameter. Specifically, each bond has a face value b_t that decays over time at rate λ, and the bond makes continuous payments $1/P_t \cdot \lambda b_t \mathrm{d}t$ of final goods to its holder.[29] We continue to assume that monetary policy controls the nominal short rate i_t.[30] In the limit $\lambda \to \infty$, this setup collapses to our baseline model.

We relegate a more detailed description of the generalized model as well as all model derivation steps to Appendix G. Here, we only present the main conclusions.

As before, let B_t be the face value of the total stock of outstanding government bonds and μ_t^B its growth rate. Let furthermore P_t^B denote the nominal bond price, so that $q_t^B K_t = P_t^B B_t / P_t$ is the real value of the bond stock. The government's flow budget

[29]Assuming a settlement in goods indexed to P_t instead of money avoids the need to also introduce money into the model, but is otherwise without loss of generality.

[30]The nominal short rate equals the nominal interest rate on a hypothetical zero-net-supply bond with infinitesimal duration.

constraint in this setting can be written as

$$\underbrace{\lambda\left(\frac{1}{P_t^B}-1\right)}_{=:\bar{i}_t}-\mu_t^B=\breve{s}_t,$$

where as before, $\breve{s}_t = \frac{\tau_t^K q_t^K}{q_t^B}$ is the surplus-to-debt ratio. The only difference to our baseline model is that nominal net payouts to bondholders are not equal to i_t, but to a variable $\bar{i}_t$ that turns out to be a type of smoothed version of future interest rates i_t.

As in our baseline model, only the difference $\bar{i}_t - \mu_t^B$ enters the return on bonds and, hence, it is again only the fiscal policy variable $\breve{s}_t$ that affects portfolio choice between bonds and capital. In fact, Propositions 2 and 3 about the determination of the bond wealth share ϑ_t and the separation of portfolio choice from nominal variables, including the policy rate i_t, remain valid here.[31]

The important change is that, while the nominal price level P_t is sticky, the nominal bond price P_t^B is not. As a consequence, the (scaled) safe asset value $q_t^B = P_t^B \frac{B_t}{P_t K_t}$ is no longer a purely backward-looking state variable. Instead, only its second factor, $\bar{q}_t^B := \frac{B_t}{P_t K_t}$, is a state. Its evolution can be written as

$$\frac{\mathrm{d}\bar{q}_t^B}{\bar{q}_t^B} = \left(\bar{i}_t - \breve{s}_t - \pi_t - g_t\right)\mathrm{d}t \tag{37}$$

in full analogy to equation (31), except that the policy instrument i_t is here replaced with $\bar{i}_t$.

Finally, the nominal bond price P_t^B can be written in terms of $\bar{i}_t$. The actual safe asset value q_t^B is then related to the state variable $\bar{q}_t^B$ by the identity

$$q_t^B = \frac{\lambda}{\lambda + \bar{i}_t}\bar{q}_t^B. \tag{38}$$

The link between the state variable $\bar{q}_t^B$ and the safe asset value q_t^B is thus solely determined by $\bar{i}_t$. The shorter is the debt duration (larger λ), the larger movements in $\bar{i}_t$ are required to move q_t^B away from $\bar{q}_t^B$ in order to stabilize aggregate demand.[32]

[31] As a corollary, the natural (flexible price) allocation is also unaffected by the bond duration parameter λ.

[32] While not the focus of this paper, the equation also shows that the impact of a lower bound on nominal interest rates for aggregate demand stabilization is more severe, the shorter is the duration of government debt: for larger λ, monetary policy needs to engineer a smaller (more negative) weighted-average interest rate $\bar{i}_t$ to offset a shortfall in aggregate demand.

In sum, all model equations are the same as in our baseline model, except that i_t is everywhere replaced with $\bar{i}_t$ and q_t^B now is the product of a backward-looking state term $\bar{q}_t^B$ and a "jump" term that only depends on $\bar{i}_t$. We show in the appendix that $\bar{i}_t$ in turn is related solely to the path of future policy rates $\{i_s\}_{s\geq t}$. Without aggregate risk in policy rates, this relationship can be represented as

$$\bar{i}_t = \int_t^\infty \left(\lambda + \bar{i}_s\right) e^{-\int_t^s \left(\lambda + \bar{i}_{s'}\right) \mathrm{d}s'} i_s \mathrm{d}s,$$

i.e. $\bar{i}_t$ is essentially a weighted average of future nominal short rates. The same intuition remains valid when interest rates are stochastic, although the formal relationship is then more complex. We can thus work directly with $\bar{i}_t$ instead of i_t as a monetary policy instrument.[33]

As discussed in Sections 3 and 5, the inability of monetary policy to stabilize aggregate demand on impact of an uncertainty shock in our baseline model is due to the fact that there $q_t^B = \bar{q}_t^B$ is a state that does not react to monetary policy. Here, monetary policy $\bar{i}_t$ can affect q_t^B directly through equation (38). Nevertheless, this is insufficient to implement the flexible price allocation (or any other desired path of q^B). The reason is that with the single instrument $\bar{i}_t$, policy cannot simultaneously implement the flexible-price safe asset value q_t^B through equation (38) to close the output gap in the short run and set the nominal rate to the natural rate to make the state evolution (37) consistent with zero inflation in the long run. This is because the implementation of each policy objective constrains the full path of future policy rates, but the requirements to achieve both objectives simultaneously are usually incompatible, except by coincidence.

To illustrate this point clearly, we consider a very simple example: suppose we start the economy in a steady state with constant $\tilde{\sigma}$ and consider an unanticipated shock that permanently raises $\tilde{\sigma}$ at $t = 0$. For unchanged fiscal policy $\breve{s}$,[34] the permanent shock leads to a (one-time) permanent increase in ϑ and the flexible-price level of q^B. For simplicity, restrict attention to a simple monetary policy that changes the nominal rate i only once to a new level and keeps it there forever (then automatically $\bar{i} = i$).[35]

[33] There are some technical restrictions on which paths for $\bar{i}$ are attainable. However, one can show that any bounded path that satisfies $\inf_{t\geq 0} \bar{i}_t > \lambda$ is always feasible.

[34] An identical conclusion would hold for a one-time change in $\breve{s}$ to a new constant value to implement the optimal new steady state policy

[35] The limited room for monetary policy in this example is *not* due to this additional assumption.

Monetary policy could to do one of the following:

(1) Set $i = i^n$ equal to the new (constant) natural rate. This policy ensures that q_t^B converges to the flexible price steady state value and there is no distortion from sticky prices in the long run. Given this policy, equation (38) then implies that the safe asset value on shock impact jumps to

$$q_0^B = \frac{\lambda}{\lambda + i^n} \bar{q}_0^B.$$

This happens to be equal to the new flexible price steady state level of q^B for only a single specific value of the parameter λ.

(2) Set i to produce a jump in q^B at $t = 0$ that moves q_0^B immediately to the new flexible price steady state $q^{B,flex}$. This implies that at the time the shock hits, there is no output gap. By equation equation (38), the required interest rate is

$$i = \lambda \left(\bar{q}_0^B / q^{B,flex} - 1 \right).$$

This required rate happens to be equal to the natural rate i^n for only a single specific value of λ. Otherwise, this policy introduces a drift into the state variable $\bar{q}_t^B$, so that both $\bar{q}^B$ and q^B moves over time. Thus q^B starts to deviate from its flexible price level in future periods and an output gap opens up then.

Except for one specific bond duration parameter λ, the policy instruments i required for (1) and (2) thus differ. Unless λ happens to be just right, policy cannot simultaneously generate zero output gaps and zero inflation.[36]

7 Conclusions

In this paper, we present a New-Keynesian model with idiosyncratic risk and nominal government bonds. We highlight a new transmission mechanism of uncertainty shocks through household portfolios. As uncertainty builds up, the households rebalance their portfolios towards government bonds, which leads to overshooting in the

Allowing $\{i_t\}$ to follow an arbitrary path would not affect the conclusion.

[36]Moreover, that "coincidence level" of debt maturity depends on the size of the shock. Generally, it is thus not possible to set λ ex ante such that (1) and (2) coincide ex post.

price of capital price when prices are sticky. The drop in the price of capital further depresses aggregate demand and results in declines in economic activities. Monetary policy alone has limited impact on household portfolio without coordinated fiscal policy. Finally, we show that the government can stabilize the economy with non-distortionary taxes and restore constrained efficiency with distortionary taxes.

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A The New-Keynesian Phillips Curve

Proposition 7 (The New-Keynesian Phillips Curve) *In a symmetric equilibrium where all firms set the same price and produce the same amount of output ($y_t^j = Y_t$, $P_t^j = P_t$, $\pi_t^j = \pi_t$), the inflation rate satisfies*

$$d\Pi_t = \left[\Pi_t \left(r_t - g_t + \sigma_t^{\Pi}\varsigma_t\right) - \frac{\varepsilon}{\kappa}(p_t^R - p^{R,flex})u_t\right] dt + \Pi_t\sigma_t^{\Pi} dZ_t. \tag{39}$$

where $r_t := i_t - \pi_t$, $\Pi_t := \pi_t \frac{Y_t}{K_t}$, $\varsigma_t = (1-\vartheta)\sigma_t^{q,K}$ and $p^{R,flex} = a\frac{\varepsilon-1}{\varepsilon}$.

Proof. The intermediary goods firms' problem implies the following Hamiltonian:

$$H_t = \frac{\xi_t}{\xi_0}\left[\left(\frac{P_t^j}{P_t} - \frac{p_t^R}{a}\right)\left(\frac{P_t^j}{P_t}\right)^{-\varepsilon} - \frac{\kappa}{2}\left(\pi_t^j\right)^2\right] Y_t + \lambda_t \pi_t^j P_t^j.$$

The optimality condition is

$$\frac{\xi_t}{\xi_0}\kappa\pi_t^j Y_t = \lambda_t P_t^j \qquad \Longrightarrow \qquad \pi_t^j = \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t^j}{Y_t}.$$

The co-state equation is (with $P_t^j = P_t$ imposed)

$$\begin{aligned} d\lambda_t &= -\frac{\partial H}{\partial P_t^j} dt + \lambda_t \sigma_t^{\lambda} dZ_t \\ &= -\left[\frac{\xi_t}{\xi_0}\frac{Y_t}{P_t}\frac{\varepsilon}{a}(p_t^R - p^{R,flex}) + \lambda_t\pi_t\right] dt + \lambda_t\sigma_t^{\lambda} dZ_t. \end{aligned}$$

Define

$$\Pi_t = a u_t \pi_t = \pi_t \frac{Y_t}{K_t} = \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t}{K_t}.$$

By Ito's Lemma,

$$
\begin{aligned}
\mathrm{d}\Pi_t &= \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{P_t}{K_t}\mathrm{d}\lambda_t + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t}{K_t}\mathrm{d}P_t + \frac{1}{\kappa}\xi_0\frac{\lambda_t P_t}{K_t}\mathrm{d}\frac{1}{\xi_t} + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\lambda_t P_t \mathrm{d}\frac{1}{K_t} + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t}{K_t}(\sigma_t^\lambda \varsigma_t)\mathrm{d}t \\
&= \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{P_t}{K_t}\left[-\left(\frac{\xi_t}{\xi_0}\frac{Y_t}{P_t}\frac{\varepsilon}{a}(p_t^R - p^{R,flex}) + \lambda_t\pi_t\right)\mathrm{d}t + \lambda_t\sigma_t^\lambda \mathrm{d}Z_t\right] + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t}{K_t}\pi_t\mathrm{d}t \\
&\qquad + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t}{K_t}\frac{\mathrm{d}(1/\xi_t)}{1/\xi_t} + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t}{K_t}\frac{\mathrm{d}(1/K_t)}{1/K_t} + \frac{1}{\kappa}\frac{\xi_0}{\xi_t}\frac{\lambda_t P_t(i)}{K_t}(\sigma_t^\lambda\varsigma_t)\mathrm{d}t \\
&= -\frac{\varepsilon}{\kappa a}(p_t^R - p^{R,flex})\frac{Y_t}{K_t}\mathrm{d}t + \Pi_t(r_t + \varsigma_t^2)\mathrm{d}t - \Pi_t g_t \mathrm{d}t + \Pi_t(\sigma_t^\lambda\varsigma_t)\mathrm{d}t + \Pi_t(\varsigma_t + \sigma_t^\lambda)\mathrm{d}Z_t \\
&= \left[\Pi_t(r_t - g_t) + \Pi_t(\varsigma_t^2 + \sigma_t^\lambda\varsigma_t) - \frac{\varepsilon}{\kappa a}(p_t^R - p^{R,flex})\frac{Y_t}{K_t}\right]\mathrm{d}t + \Pi_t(\varsigma_t + \sigma_t^\lambda)\mathrm{d}Z_t
\end{aligned}
$$

Note that we used

$$
\begin{aligned}
\frac{\mathrm{d}\xi_t}{\xi_t} &= -r_t\mathrm{d}t - \varsigma_t\mathrm{d}Z_t \quad \Longrightarrow \quad \frac{\mathrm{d}(1/\xi_t)}{1/\xi_t} = (r_t + \varsigma_t^2)\mathrm{d}t + \varsigma_t\mathrm{d}Z_t \\
\frac{\mathrm{d}K_t}{K_t} &= g_t\mathrm{d}t \quad \Longrightarrow \quad \frac{\mathrm{d}(1/K_t)}{1/K_t} = -g_t\mathrm{d}t
\end{aligned}
$$

where r_t is the risk-free rate, ς_t is the price of aggregate risk,[37] and $g_t = \Phi(\iota_t) - \delta(u_t)$ is the growth rate of capital. Define

$$
\sigma_t^\Pi = \varsigma_t + \sigma_t^\lambda.
$$

Now we end up with equation (39). Further, notice that under log utility,

$$
\varsigma_t = (1 - \vartheta_t)\sigma_t^{q,K}.
$$

B Model Solution

We solve the model using a recursive approach. Let $V_t(n_t)$ be the households' value function. For expository purposes, we dropped the aggregate states and instead let the value function be dependent on time t. The households' problem yields the following Hamilton-Jacobi-Bellman (HJB) equation

$$
\rho V_t(n_t) = \max_{c_t, u_t, \iota_t \theta_t}\left\{\log c_t + \frac{\partial V_t(n_t)}{\partial n_t}n_t\mu_t^n + \frac{1}{2}\frac{\partial^2 V_t(n_t)}{\partial n_t^2}n_t^2\left[(\sigma_t^n)^2 + (\tilde{\sigma}_t^n)^2\right]\right\} + \frac{\mathrm{d}\mathbb{E}_t\left[V_t(n_t)\right]}{\mathrm{d}t},
$$

[37] For derivations of the dynamics of SDF, see Brunnermeier and Sannikov (2016) for example.

where μ_t^n and $\sigma_t^n, \tilde{\sigma}_t^n$ are the drift and volatility of $\mathrm{d}n_t/n_t$, respectively,

$$\mu_t^n = \theta_t(\mathrm{d}R_t^B/\mathrm{d}t) + (1-\theta_t)\mathbb{E}_t\left[\mathrm{d}R_t^K\right]/\mathrm{d}t - \frac{c_t}{n_t},$$
$$\sigma_t^n = (1-\theta_t)\sigma_t^{q,K},$$
$$\tilde{\sigma}_t^n = (1-\theta_t)\tilde{\sigma}_t.$$

The assumption of logarithmic preferences yields the convenient property of scale invariance. Following Brunnermeier and Sannikov (2016) and Di Tella and Hall (2022), we guess and verify that the households' value function takes the form

$$V_t(n_t) = A_t + \frac{\log n_t}{\rho}, \tag{40}$$

where A_t denotes investment opportunities of the households and is potentially a function of the aggregate states $\{q_t^B, \tilde{\sigma}_t\}$. We denote

$$\mathrm{d}A_t = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t.$$

Plugging the guess into the HJB equation,

$$\rho A_t + \log n_t = \max_{c_t, u_t, \iota_t \theta_t}\left\{\log c_t + \frac{\mu_t^n}{\rho} - \frac{1}{2}\frac{(\sigma_t^n)^2 + (\tilde{\sigma}_t^n)^2}{\rho}\right\} + \mu_t^A. \tag{41}$$

Taking the first-order conditions, we see that $c_t = \rho n_t$ and that the choice of u_t, ι_t, θ_t do not depend on n_t. Further, since in equilibrium all households choose the same portfolio θ_t and consumption rate $c_t/n_t = \rho$, we see that μ_t^n and σ_t^n are also independent from the individual state n_t. Hence, $\log n_t$ can be cancelled out on both sides of (41). In addition, A_t must satisfy

$$\mu_t^A = \rho A_t - \max_{c_t, u_t, \iota_t \theta_t}\left\{\log \rho + \frac{\mu_t^n}{\rho} - \frac{1}{2}\frac{(\sigma_t^n)^2 + (\tilde{\sigma}_t^n)^2}{\rho}\right\}. \tag{42}$$

The transversality condition requires that $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}\left[A_T\right] = 0$, which ensures that the linear backward stochastic differential equation (BSDE) (42) has a unique solution. Therefore, we conclude that (40) is indeed the household value function.

C The Numerical Algorithm

Here we outline the numerical algorithm used to solve the model in Section 2.4. First, from the market clearing equation (21),

$$\rho(q_t^B + q_t^K) = u_t a - \iota_t \quad \Longrightarrow \quad \frac{u_t a - \iota_t}{q_t^K} = \frac{\rho}{1-\vartheta_t} \quad \Longrightarrow \quad q_t^K = (1-\vartheta_t)\frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho}. \tag{43}$$

Using (19), we can then express p_t^R as

$$p_t^R = \frac{1-\vartheta_t}{1-\vartheta_t+\phi\rho}(1+\phi a u_t)\delta'(u_t). \tag{44}$$

We can rewrite the bond evaluation equation (23), the consistency requirement (15), and the Phillips curve (39) as

$$\mu_t^{\vartheta} = \rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \breve{s}_t \tag{45}$$

$$\mu_t^{q,B} = \mu_t^B - \pi_t - \Phi(\iota_t) + \delta(u_t) \tag{46}$$

$$\mu_t^{\Pi} = \left[i_t - \pi_t - \Phi(\iota_t) + \delta(u_t) + \sigma_t^{\Pi}\varsigma_t\right] - \frac{\varepsilon}{\kappa}\left[\frac{1-\vartheta_t}{1-\vartheta_t+\phi\rho}(1+\phi a u_t)\delta'(u_t) - p^{R,flex}\right]\frac{u_t}{\Pi_t} \tag{47}$$

where

$$u_t = \frac{1}{a}\left(\rho(q_t^K + q_t^B) + \frac{1}{\phi}(q_t^K - 1)\right) = \frac{1}{a}\left[\left(\rho + \frac{1}{\phi}\right)\frac{q_t^B}{\vartheta_t} - \frac{1}{\phi}(1+q_t^B)\right]$$

$$\varsigma_t = (1-\vartheta_t)\sigma_t^{q,K}$$

$$\sigma_t^{q,K} = -\frac{\vartheta'(\tilde{\sigma}_t)\nu(\tilde{\sigma}_t)}{\vartheta_t(1-\vartheta_t)}$$

Postulate that

$$\vartheta_t = \vartheta(t, q_t^B, \tilde{\sigma}_t), \quad \Pi_t = \Pi(t, q_t^B, \tilde{\sigma}_t).$$

For generality, the dynamics of $\tilde{\sigma}_t$ is denoted by

$$\mathrm{d}\tilde{\sigma}_t = \mu(\tilde{\sigma}_t)\mathrm{d}t + \nu(\tilde{\sigma}_t)\mathrm{d}Z_t.$$

Then, by Ito's lemma,

$$\begin{aligned}
\vartheta\mu^{\vartheta} &= \partial_t\vartheta + (\partial_q\vartheta)q^B\mu^{q,B} + (\partial_\sigma\vartheta)\mu(\tilde{\sigma}) + \frac{1}{2}(\partial_{\sigma\sigma}\vartheta)(\nu(\tilde{\sigma}))^2, \\
\Pi\mu^{\Pi} &= \partial_t\Pi + (\partial_q\Pi)q^B\mu^{q,B} + (\partial_\sigma\Pi)\mu(\tilde{\sigma}) + \frac{1}{2}(\partial_{\sigma\sigma}\Pi)(\nu(\tilde{\sigma}))^2 \\
\Pi\sigma^{\Pi} &= (\partial_\sigma\Pi)\nu(\tilde{\sigma})
\end{aligned}$$

Plugging the expressions above into (45) - (47), we get

$$\begin{aligned}
\partial_t\vartheta = \vartheta &\left[\rho - (1-\vartheta)^2\tilde{\sigma}^2 - \breve{s}\right] \\
&- (\partial_q\vartheta)q^B\left[\mu^B - \frac{\Pi}{au} - \Phi(\iota) + \delta(u)\right] - (\partial_\sigma\vartheta)\mu(\tilde{\sigma}) - \frac{1}{2}(\partial_{\sigma\sigma}\vartheta)(\nu(\tilde{\sigma}))^2 \\
\partial_t\Pi = \Pi &\left[i - \frac{\Pi}{au} - \Phi(\iota) + \delta(u) + \sigma^{\Pi}\varsigma\right] \\
&- \frac{\varepsilon}{\kappa}\left[\frac{1-\vartheta}{1-\vartheta+\phi\rho}(1+\phi au)\delta'(u) - a\frac{\varepsilon-1}{\varepsilon}\right]u \\
&- (\partial_q\Pi)q^B\left[\mu^B - \frac{\Pi}{au} - \Phi(\iota) + \delta(u)\right] - (\partial_\sigma\pi)\mu(\tilde{\sigma}) - \frac{1}{2}(\partial_{\sigma\sigma}\pi)(\nu(\tilde{\sigma}))^2
\end{aligned}$$

where

$$\begin{aligned}
\iota &= \frac{1}{\phi}\left(q^B\frac{1-\vartheta}{\vartheta} - 1\right) & \sigma^{\Pi} &= (\partial_\sigma\Pi)\frac{\nu(\tilde{\sigma})}{\Pi} \\
u &= \frac{1}{a}\left[\left(\rho + \frac{1}{\phi}\right)\frac{q^B}{\vartheta} - \frac{1}{\phi}(1+q^B)\right] & i &= i(q^B,\tilde{\sigma}) \\
\sigma^{q,K} &= -\frac{(\partial_\sigma\vartheta)\nu(\tilde{\sigma})}{\vartheta(1-\vartheta)} & \breve{s} &= \breve{s}(q^B,\tilde{\sigma}) \\
\varsigma &= (1-\vartheta)\sigma^{q,K}
\end{aligned}$$

To solve this system of PDE, we start with guesses for terminal conditions $\vartheta(T,q^B,\tilde{\sigma}), \pi(T,q^B,\tilde{\sigma})$ and iterate backwards in time using a finite difference method.

D Flexible Price Equilibrium

Portfolio Choice. Assume that the excess debt growth is a function of $(\tilde{\sigma}_t,\vartheta_t)$, $\breve{s}_t = \breve{s}(\tilde{\sigma}_t,\vartheta_t)$. We postulate that $\vartheta_t = \vartheta(\tilde{\sigma}_t)$. By Ito's Lemma,

$$\mu_t^{\vartheta} = \vartheta'(\tilde{\sigma}_t)\frac{\mu(\sigma_t)}{\vartheta_t} + \frac{1}{2}\vartheta''(\tilde{\sigma}_t)\frac{[\nu(\sigma_t)]^2}{\vartheta_t}, \qquad \sigma_t^{\vartheta} = \vartheta'(\tilde{\sigma}_t)\frac{\nu(\tilde{\sigma}_t)}{\vartheta_t}.$$

Then, ϑ_t can be solved from the bond evaluation equation

$$\mu_t^{\vartheta} = \rho - (1 - \vartheta(\tilde{\sigma}_t))^2 \tilde{\sigma}_t^2 - \breve{s}(\tilde{\sigma}_t, \vartheta(\tilde{\sigma}_t))$$

$$\vartheta'(\tilde{\sigma}_t) \frac{\mu(\sigma_t)}{\vartheta_t} + \frac{1}{2} \vartheta''(\tilde{\sigma}_t) \frac{[\nu(\sigma_t)]^2}{\vartheta_t} = \rho - (1 - \vartheta(\tilde{\sigma}_t))^2 \tilde{\sigma}_t^2 - \breve{s}(\tilde{\sigma}_t, \vartheta(\tilde{\sigma}_t))$$

which reduces to a one-dimensional ordinary differential equation.

Production and Investment. When prices are flexible, the intermediary goods firms' problem collapses to a static profit-maximization problem at each time

$$\max_{P_t^j} \left(\frac{P_t^j}{P_t} - \frac{p_t^R}{a} \right) y_t^j \qquad \text{s.t.} \qquad y_t^j = \left(\frac{P_t^j}{P_t} \right)^{-\varepsilon} Y_t.$$

The firms set their prices at

$$\frac{P_t^j}{P_t} = \frac{1}{a} \frac{\varepsilon}{\varepsilon - 1} p_t^R.$$

In a symmetric equilibrium all firms charge the same price ($P_t^j = P_t$), so

$$p_t^{R,flex} = a \frac{\varepsilon - 1}{\varepsilon}. \tag{48}$$

First, from equation (44) and (48), we see that equilibrium utilization is a function of the portfolio weight ϑ_t,

$$a \frac{\varepsilon - 1}{\varepsilon} = \frac{1 - \vartheta_t}{1 - \vartheta_t + \phi\rho} (1 + \phi a u_t) \delta'(u_t) \tag{49}$$

As we did in equation (43), we can express q_t^K, q_t^B, ι_t in terms of ϑ,[38]

$$q_t^K = (1 - \vartheta_t) \frac{1 + \phi a u_t}{1 - \vartheta_t + \phi\rho},$$

$$q_t^B = \frac{\vartheta_t}{1 - \vartheta_t} q_t^K = \vartheta_t \frac{1 + \phi a u_t}{1 - \vartheta_t + \phi\rho},$$

$$\iota_t = a u_t - \rho(q_t^K + q_t^B) = \frac{a u_t (1 - \vartheta_t) - \rho}{1 - \vartheta_t + \phi\rho}. \tag{50}$$

The steady-state capital growth rate can be backed out from equations (49) and (50)

$$g_t^n = \Phi(\iota_t) - \delta(u_t).$$

[38]When $\Phi(\iota) = \log(1 + \phi\iota)/\phi$ and $\delta(u) = (\bar{\delta}/2)u^2$, the flexible price values of q_t^K and q_t^B are given by equations (27) and (28).

The Model without a Safe Asset. The flexible price allocation of the model is similar. Specifically, since $\vartheta_t = 0$, equations (49) and (50) becomes

$$a\frac{\varepsilon-1}{\varepsilon} = \frac{1}{1+\phi\rho}(1+\phi a u_t)\delta'(u_t), \quad \iota_t = \frac{a u_t - \rho}{1+\phi\rho}.$$

The steady-state capital growth rate is still $g_t^n = \Phi(\iota_t) - \delta(u_t)$.

E Optimal Policy Implementation

Suppose that the government imposes a subsidy τ_t^f on intermediary goods firms.[39] In this case, the rental price of capital under flexible prices (26) is modified to

$$p^{R,flex} = (1+\tau_t^f)a\frac{\varepsilon-1}{\varepsilon}.$$

Together with (44), we see that utilization under flexible prices is determined by

$$(1+\tau_t^f)a\frac{\varepsilon-1}{\varepsilon} = \frac{1-\vartheta_t}{1-\vartheta_t+\phi\rho}(1+\phi a u_t)\delta'(u_t).$$

We then get the optimal τ_t^f by plugging in the efficient level of portfolio weight ϑ_t^o and utilization rate u_t^o from the planner's problem (36).

In additional, the first-order conditions to the planner's problem are

$$\delta'(u_t) = \frac{a}{1+\phi a u_t}\left[\rho\phi + (1-\vartheta_t) + \frac{1}{\rho}(1-\vartheta_t+\phi\rho)(1-\vartheta_t)^2\tilde{\sigma}_t^2\right],$$
$$\Phi'(\iota_t) = \frac{1-\vartheta_t+\phi\rho}{1+\kappa a u_t}\left[1+\frac{1}{\rho}(1-\vartheta_t+\phi\rho)(1-\vartheta_t)\tilde{\sigma}_t^2\right].$$

Simplifying the first-order conditions, we see that the optimal ϑ_t^o satisfies the following cubic equation,

$$\vartheta_t^3 - (3+\phi\rho)\vartheta_t^2 + \left(3+2\rho\phi+\frac{\rho}{\tilde{\sigma}_t^2}\right)\vartheta_t - (1+\phi\rho) = 0.$$

[39] For simplicity, we assume that the subsidy is funded by lump-sum taxes on the firms, so it is a pure decision wedge and does not interact with household portfolio choice.

Applying Ito's Lemma gives us the dynamics of ϑ_t^o,

$$\sigma_t^{\vartheta} = \frac{2\rho\nu(\tilde{\sigma}_t)}{\tilde{\sigma}_t^3\left[3\vartheta_t^2 - 2(3+\phi\rho)\vartheta_t + (3+2\rho\phi+\frac{\rho}{\tilde{\sigma}_t^2})\right]},$$

$$\mu_t^{\vartheta} = \frac{\frac{2\rho}{\tilde{\sigma}^3}\mu(\tilde{\sigma}) - \frac{3\rho}{\tilde{\sigma}^4}[\nu(\tilde{\sigma})]^2 + \frac{2\rho}{\tilde{\sigma}^3}\sigma_t^{\vartheta}\nu(\tilde{\sigma}) - (\sigma_t^{\vartheta})^2\left[3\vartheta_t^2 - (3+\phi\rho)\vartheta_t\right]}{3\vartheta_t^2 - 2(3+\phi\rho)\vartheta_t + (3+2\rho\phi+\frac{\rho}{\tilde{\sigma}_t^2})}.$$

Recall the bond evaluation equation

$$\mu_t^{\vartheta} = \rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \check{s}_t.$$

It is then obvious that the optimal tax rate should be

$$\check{s}_t = \rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \mu_t^{\vartheta},$$
$$\Longrightarrow \quad \tau_t^K = \frac{\vartheta_t}{1-\vartheta_t}\left[\rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \mu_t^{\vartheta}\right].$$

F Proofs

Proof of Proposition 1. First, since the households are homogeneous and the intermediary goods firms are symmetric, the transfers from firms (7) are simply

$$\omega_t = au_t - p_t^R u_t.$$

Hence, the return on capital (5) is

$$dR_t^{K,i}(u_t^i, \iota_t^i) = \left(\frac{au_t^i - \iota_t^i}{q_t^K} - \tau_t^K + g(u_t^i, \iota_t^i) + \mu_t^{q,K}\right)dt + \tilde{\sigma}_t d\tilde{Z}_t^i + \sigma_t^{q,K} dZ_t.$$

Then we can easily obtain the first-order conditions from the HJB laid out in Appendix B (i.e., equation (41)).

Proof of Proposition 2. Plugging the expressions for dR_t^B and dR_t^K into the first-order condition (20), we get

$$\frac{au_t - \iota_t}{q_t^K} - \tau_t^K + g_t + \mu_t^{q,K} - \left(i_t - \mu_t^B + g_t - \mu_t^{q,B}\right) = (1-\vartheta_t)\left[\tilde{\sigma}_t^2 + (\sigma_t^{q,K})^2\right]$$

$$\frac{au_t - \iota_t}{q_t^K} - (1-\vartheta_t)\tilde{\sigma}_t^2 - (i_t - \mu_t^B + \tau_t^K) = \mu_t^{q,B} - \mu_t^{q,K} + (1-\vartheta_t)(\sigma_t^{q,K})^2. \qquad (51)$$

Applying Ito's Lemma to the asset market clearing condition $\vartheta_t = \frac{q_t^B}{q_t^K + q_t^B}$ yields

$$\sigma_t^{q,B} - \sigma_t^{q,K} = \frac{\sigma_t^{\vartheta}}{1-\vartheta_t},$$

$$\mu_t^{q,B} - \mu_t^{q,K} = \frac{1}{1-\vartheta_t}\left[\mu_t^{\vartheta} + \sigma_t^{\vartheta}\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right)\right].$$

Since $\sigma_t^{q,B} = 0$,

$$\mu_t^{q,B} - \mu_t^{q,K} + (1-\vartheta_t)(\sigma_t^{q,K})^2 = \frac{1}{1-\vartheta_t}\mu_t^{\vartheta}.$$

Further, notice that the goods market clearing condition implies that

$$au_t - \iota_t = \rho(q_t^B + q_t^K) \quad \Longrightarrow \quad \frac{au_t - \iota_t}{q_t^K} = \frac{\rho}{1-\vartheta_t}.$$

Finally, the government budget constraint (9) requires that

$$i_t - \mu_t^B + \tau_t^K = \frac{1}{1-\vartheta_t}\check{s}_t.$$

Plugging all terms into (51) results in the bond evaluation equation (23). Applying Ito's Lemma to $(e^{-\rho t}\vartheta_t)$ and integrating forward gives us the integral form (24).

Proof of Proposition 3. ϑ_t is given by the bond evaluation equation

$$\vartheta_t = \mathbb{E}_t\left\{\int_t^{\infty} e^{-\rho(s-t)}\vartheta_s\left[(1-\vartheta_s)^2\tilde{\sigma}_s^2 + \check{s}(\tilde{\sigma}_t, \vartheta_t)\right]\mathrm{d}s\right\}.$$

It is obvious that the solution to this equation only depends on $\tilde{\sigma}_t$.

Proof of Proposition 4. Consider equation (34),

$$\frac{u_t a - \iota_t}{q_t^K} + g_t + \left[\mu_t^{q,K} - (\sigma_t^{q,K})^2\right] = r_t^f + \tau_t^K + \tilde{\sigma}_t^2.$$

We want an equilibrium with $\pi_t = \mu_t^{q,K} = \mu_t^{q,B} = \tau_t^K = 0$ and $r_t^f = i_t$, so

$$i_t = \frac{u_t a - \iota_t}{q_t^K} + g_t - \tilde{\sigma}_t^2.$$

By market clearing, $(u_t a - \iota)/q_t^K = \rho$. Then the capital growth rate coincides with its steady-state level if the interest rate is set at

$$i_t^n = \rho + g_t^n - \tilde{\sigma}_t^2.$$

By standard arguments, we need other equilibrium selection techniques to ensure that this is the unique equilibrium (e.g., the Taylor Principle).

Proof of Proposition 5. With lump-sum taxes, the government budget constraint is

$$i_t B_t = \mu_t^B B_t + P_t \tau_t K_t + P_t \tau_t^K \cdot q_t^K K_t \quad \Longrightarrow \quad i_t = \mu_t^B + \frac{1}{q_t^B}\tau_t + \frac{q_t^K}{q_t^B}\tau_t^K.$$

The consistency condition (15) becomes

$$\mu_t^{q,B} = \mu_t^B - \pi_t - g_t = i_t - \frac{\tau_t}{q_t^B} - \frac{q_t^K}{q_t^B}\tau_t^K - \pi_t - g_t. \tag{52}$$

The return on bonds is now

$$\mathrm{d}R_t^B = i_t \mathrm{d}t + \frac{\mathrm{d}(q_t^B K_t / B_t)}{q_t^B K_t / B_t} = \left(\frac{1}{q_t^B}\tau_t + \frac{q_t^K}{q_t^B}\tau_t^K + g_t + \mu_t^{q,B} \right) \mathrm{d}t.$$

No arbitrage between government bonds and tax claims requires

$$\frac{\tau_t}{q_t^\tau} + \mu_t^{q,\tau} = \frac{\tau_t}{q_t^B} + \frac{q_t^K}{q_t^B}\tau_t^K + \mu_t^{q,B} \tag{53}$$

The total supply of idiosyncratic-risk-free assets to the household sector is $q_t^G K_t = (q_t^B - q_t^\tau) K_t$. The portfolio of all safe assets that is long in the bond stock and short in the total tax liabilities has the return

$$\mathrm{d}R_t^G = \frac{q_t^B}{q_t^G}\mathrm{d}R_t^B - \frac{q_t^\tau}{q_t^G}\mathrm{d}R_t^\tau = \left(\frac{q_t^K}{q_t^G}\tau_t^K + g_t + \mu_t^{q,G} \right) \mathrm{d}t. \tag{54}$$

In addition, the law of motion of q_t^G is given by

$$\mu_t^{q,G} = \frac{q_t^B}{q_t^G}\mu_t^{q,B} - \frac{q_t^\tau}{q_t^G}\mu_t^{q,\tau} = \left(i_t - \frac{q_t^K}{q_t^G}\tau_t^K \right) - \pi_t - g_t. \tag{55}$$

Comparing (54) with (4) and (9), we see that the return $\mathrm{d}R_t^G$ is the same as the return on bonds $\mathrm{d}R_t^B$ in the model without lump-sum taxes. Similar, the law of motion of q_t^G (55) is the same as (15), the law of motion of q_t^B in the baseline model. In equilibrium, market clearing dictates that the households collectively hold all safe assets (long in government bonds and short in tax claims). It is then obvious that the economy with lump-sum taxes has the same dynamics as before with the price of bonds q_t^B replaced by the price of all safe assets q_t^G.

Using equations (52) and (53), we see that in the model with lump-sum taxes, q_t^τ satisfies the following ODE

$$\frac{\mathrm{d}q_t^\tau}{q_t^\tau} = \left(i_t - \pi_t - g_t - \frac{\tau_t}{q_t^\tau} \right) \mathrm{d}t. \tag{56}$$

We consider a zero-inflation equilibrium with $\pi_t = 0$. Suppose that in the flexible price equilibrium, the nominal rate tracks the economic growth rate $i_t = g_t^n$. In the equilibrium, the ODE (56) then becomes

$$\frac{\mathrm{d}q_t^\tau}{\mathrm{d}t} = -\tau_t,$$

which is a forward equation with solutions of the following form

$$q_t^\tau = \int_t^\infty \tau_s \mathrm{d}s + D.$$

where D is a constant. For simplicity, we assume that there exist no bubbles in the valuation of tax claims, so $D = 0$.

For a given value of q_t^B, the government can then choose $\{\tau_s\}_{s=t}^\infty$ to ensure that the total value of government liability is the same as the bond price under flexible prices,

$$q_t^G = q_t^B - q_t^\tau = q_t^{B,flex}.$$

As we have argued in Proposition 3, the household portfolio problem is unaffected by price-setting frictions, so q_t^K will be at its flexible price level whenever q_t^G is.

Proof of Proposition 6. The welfare decomposition follows directly from the consumption decomposition in the main text. For the second integral, we use the mathematical result below.

For an Ito process X_t where $\mathrm{d}X_t/X_t = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t$, the dynamics of $\log X_t$ is given by

$$\mathrm{d}\log X_t = \left[\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2 \right] \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t.$$

Hence,

$$\log X_t = \log X_0 + \int_0^t \left[\mu_s^X - \frac{1}{2}(\sigma_s^X)^2 - \frac{1}{2}(\tilde{\sigma}_s^X)^2 \right] \mathrm{d}s + \underbrace{\int_0^t \sigma_s^X \mathrm{d}Z_s + \int_0^t \tilde{\sigma}_s^X \mathrm{d}\tilde{Z}_s}_{\text{mean zero}}.$$

Using integration by parts, one can show that

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log X_t \mathrm{d}t\right] = \frac{1}{\rho}\log X_0 + \frac{1}{\rho}\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right)\mathrm{d}t\right].$$

G Additional Details on Model Extension with Long-term Bonds

We start by deriving the return on bonds $\mathrm{d}R_t^B$ in the model with long-term bonds that replaces equation (4) from the baseline model.

Suppose the nominal bond price P_t^B satisfies

$$\frac{\mathrm{d}P_t^B}{P_t^B} = \mu_t^{P,B}\mathrm{d}t + \sigma_t^{P,B}\mathrm{d}Z_t.$$

Then the return on bonds is (in general, permitting flexible prices)

$$\begin{aligned}\mathrm{d}R_t^B &= \frac{\mathrm{d}\left(\frac{P_t^B}{P_t}\right) + \lambda\left(\frac{1}{P_t} - \frac{P_t^B}{P_t}\right)\mathrm{d}t}{\frac{P_t^B}{P_t}} \\ &= \left(\mu_t^{P,B} - \pi_t - \sigma_t^P\left(\sigma_t^{P,B} - \sigma_t^P\right)\right)\mathrm{d}t + \left(\sigma_t^{P,B} - \sigma_t^P\right)\mathrm{d}Z_t + \underbrace{\lambda\left(\frac{1}{P_t^B} - 1\right)}_{=\bar{i}_t}\mathrm{d}t. \qquad (57)\end{aligned}$$

As before, we define $q_t^B := \frac{P_t^B B_t}{P_t K_t}$, so that $\frac{P_t^B}{P_t} = \frac{q_t^B K_t}{B_t}$ and thus

$$\mu_t^{P,B} - \pi_t - \sigma_t^P\left(\sigma_t^{P,B} - \sigma_t^P\right) = \mu_t^{q,B} + g_t - \mu_t^B, \qquad \sigma_t^{P,B} - \sigma_t^P = \sigma_t^{q,B}.$$

Substituting these equations into the previous return equation implies

$$\begin{aligned}\mathrm{d}R_t^B &= \left(\mu_t^{q,B} + g_t + \bar{i}_t - \mu_t^B\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t \\ &= \left(\mu_t^{q,B} + g_t + \check{s}_t\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t,\end{aligned}$$

where the second line follows from the government budget constraint. This is precisely the same equation as in the baseline model with short-term bonds.[40]

[40] To see this, compare the equation with equation (4) and replace their $i_t - \mu_t^B$ by $\check{s}_t$ using equation (11).

Clearly, the capital return (5) is also unaffected by the introduction of long-term bonds. Consequently, the portfolio choice condition between bonds and capital for each individual agent must be identical as in the baseline model. Therefore, we obtain the same bond valuation equation as before for the bond wealth share ϑ_t (equation (23)). Also, the separation result from Proposition 3 remains valid.

As a corollary, we obtain immediately that under flexible prices, the equilibrium dynamics of q_t^B, q_t^K, u_t, and ι_t and the consumption allocation across individual agents remain unaffected by the maturity structure λ and are as in the baseline model.

The derivations made in the main text and the claim that all model equations are as in the baseline model, except that i_t must be replaced with $\bar{i}_t$ and q_t^B has to be split according to equation (38) do not require further clarification here. It is thus only left to clarify the relationship between nominal short rates i_t and the variable $\bar{i}_t$.

Specifically, we show that for given ϑ dynamics (due to portfolio separation, Proposition 3), there is in equilibrium a mapping from interest rate paths $\{i_t\}_{t=0}^{\infty}$ to paths $\{\bar{i}_t\}_{t=0}^{\infty}$ of the variable $\bar{i}$ that satisfy in addition $\inf_{t\geq 0} \bar{i}_t > -\lambda$.[41] The relationship between the two is described by the backward stochastic differential equation (BSDE)

$$\mathbb{E}_t \left[\mathrm{d}\bar{i}_t\right] = \Big(\left(\lambda + \bar{i}_t\right)\left(\bar{i}_t - i_t\right) - \sigma_t^{\vartheta}\sigma_{\bar{i},t}\Big)\,\mathrm{d}t. \tag{58}$$

Here, $\sigma_{\bar{i},t}$ denotes the arithmetic volatility of $\bar{i}_t$.

Before deriving this BSDE, we show how to invert the mapping, i.e. start with some desired Ito process for $\bar{i}_t$,

$$\mathrm{d}\bar{i}_t = \mu_{\bar{i},t}\mathrm{d}t + \sigma_{\bar{i},t}\mathrm{d}Z_t$$

and back out the required process for i_t to implement it. This is important, as only then it is feasible to treat $\bar{i}_t$ as the policy instrument. Given the desired Ito process for $\bar{i}_t$, we equate $\mu_{\bar{i},t}\mathrm{d}t$ with $\mathbb{E}_t\left[\mathrm{d}\bar{i}_t\right]$ in the BSDE and obtain, after solving for i_t,

$$i_t = \bar{i}_t - \frac{\mu_{\bar{i},t} + \sigma_t^{\vartheta}\sigma_{\bar{i},t}}{\lambda + \bar{i}_t}.$$

Under the assumption $\inf_{t\geq 0} \bar{i}_t > -\lambda$, the denominator in the second term is always positive and thus i_t required to implement $\bar{i}_t$ is well-defined.

[41] This additional restriction is natural as it is equivalent to bounded nominal bond prices P_t^B (compare equation (38)).

To derive BSDE (58), consider the portfolio choice between (long-term) government bonds and a zero-net-supply nominal bond with infinitesimal duration. Call the latter M (for money), its return is (under sticky prices, so $\sigma_t^P = 0$)

$$\mathrm{d}R_t^M = i_t \mathrm{d}t + \frac{\mathrm{d}\left(1/P_t\right)}{1/P_t} = \left(i_t - \pi_t\right)\mathrm{d}t.$$

Using the return representation (57) for bonds and the fact $\sigma_t^P = 0$ under sticky prices to simplify that representation, the portfolio choice condition between bonds and money becomes

$$\bar{i}_t - i_t + \mu_t^{P,B} = \sigma_t^n \sigma_t^{P,B}.$$

Now, $\sigma_t^n = \sigma_t^{q^B+q^K} = \sigma_t^{q,B} - \sigma_t^{\vartheta} = \sigma_t^{P,B} - \sigma_t^{\vartheta}$ and $\bar{i}_t = \lambda\left(1/P_t^B - 1\right)$, so

$$\begin{aligned}\mathrm{d}\bar{i}_t &= \lambda \mathrm{d}\left(1/P_t^B\right) = \lambda \frac{1}{P_t^B}\left(-\mu_t^{P,B} + \left(\sigma_t^{P,B}\right)^2\right)\mathrm{d}t - \lambda \frac{1}{P_t^B}\sigma_t^{P,B}\mathrm{d}Z_t \\ &= \lambda \frac{1}{P_t^B}\left(\left(\bar{i}_t - i_t\right) + \sigma_t^{\vartheta}\sigma_t^{P,B}\right)\mathrm{d}t - \lambda \frac{1}{P_t^B}\sigma_t^{P,B}\mathrm{d}Z_t\end{aligned}$$

which immediately implies the BSDE (58) stated above.

We conclude this appendix by remarking that the implicit integral formula for $\bar{i}_t$ for the case without risk ($\sigma_{\bar{i},t} = 0$) stated in the main text is easily verified by taking the time derivative in this formula and comparing the result with BSDE (58).